POSITION ANALYSIS OF PLANAR TENSEGRITY STRUCTURES

By

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A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

2006

Report Documentation Page		*Form Approved* *OMB No. 0704-0188*

Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.

1. REPORT DATE **2006**	2. REPORT TYPE	3. DATES COVERED **00-00-2006 to 00-00-2006**

4. TITLE AND SUBTITLE **Position Analysis of Planar Tensegrity Structures**	5a. CONTRACT NUMBER
	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **Center for Intelligent Machines and Robotics,Department of Mechanical Engineering,University of Florida,Gainesville,FL,32611**	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT **Approved for public release; distribution unlimited**

13. SUPPLEMENTARY NOTES **The original document contains color images.**

14. ABSTRACT

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES **107**	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**			

To the scientific community.

ACKNOWLEDGMENTS

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POSITION ANALYSIS OF PLANAR TENSEGRITY STRUCTURES

By

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August 2006

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Tensegrity is an abbreviation of tension and integrity. Tensegrity structures are
spatial structures formed by a combination of rigid elements in compression (struts) and
connecting elements that are in tension (ties). In three-dimensional tensegrity structures
no pair of struts touches, and the end of each strut is connected to non-coplanar ties,
which are in tension. In two-dimensional tensegrity structures, struts still do not touch. A
tensegrity structure stands by itself in its equilibrium position and maintains its form
solely because of the arrangement of its struts and ties. The potential energy of the
system stored in the springs is at a minimum in the equilibrium position when no external
force or torque is applied. A closed-form solution of a two-spring, three-spring, and four-
spring planar tensegrity mechanism was developed to determine all possible equilibrium
configurations when no external force or moment is applied. Here "closed form" means
that all solution equilibrium poses will be determined, although for each case a high-
degree polynomial will have to be solved numerically.

CHAPTER 1
LITERATURE REVIEW

We examined the literature related to tensegrity systems, self-deployable tensegrity systems, and parallel mechanisms with compliant elements.

Knight et al. [1] reported on the line geometries of a family of tensegrity structures called skew prisms (anti-prisms, tensegrity prisms) with pairs of triangles, squares, pentagonals, hexagonals, and so on, located at tops and bases. They used the quality index with S-P-S connectors (S is a ball-and-socket joint, and P is a sliding joint). The quality index measures the geometric stability of in-parallel devices (which in three dimensions consist of a pair of rigid platforms connected by legs that are kinematical S-P-S connectors). Geometric stability depends on the geometry of the legs.

Figures 1-1 A and 1-1 C show a sequence of parallel prisms with parallel ties of fixed length.
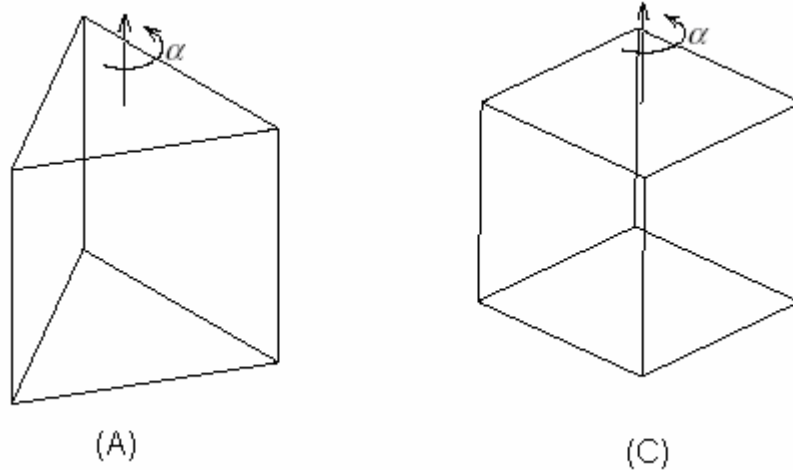
Figure 1-1. Parallel prisms.

A relative anti-clockwise rotation can be achieved by rotating the top of each prism through its vertical axes as shown in Figures 1-2 B and 1-2 D to yield a corresponding right-handed tensegrity prism that is completed by inserting struts on the diagonals of the skew quadrilaterals. Left-handed tensegrity prisms are mirror images of right-handed prisms and are obtained by rotating the tops in a clockwise direction and interchanging the ties and struts. The angle α is unique for each tensegrity prism ($\alpha = 90 - \dfrac{180}{n}$) where n is the total number of sides in the upper or lower polygons. The value of α (together with the size of the tops and bases and their distance apart) enables one to compute the length of the struts and ties that defines a unique configuration for the tensegrity prism.

(B)   (D)
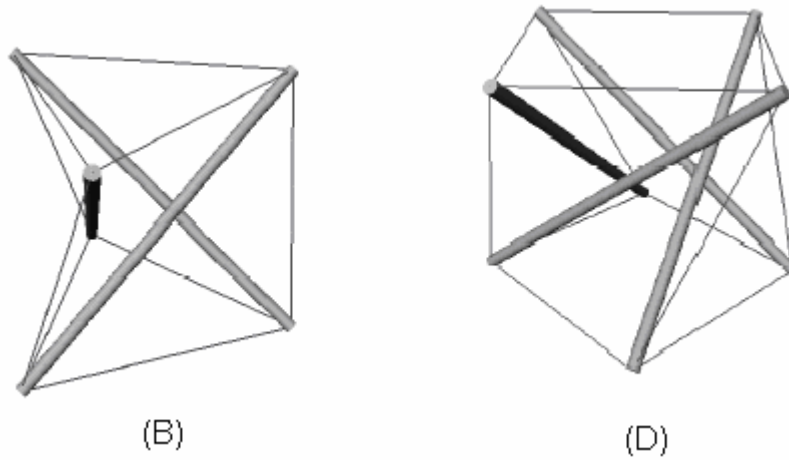
Figure 1-2. Tensegrity prisms.

The quality index for each skew prism is zero, which means that it has instantaneous mobility. Further examination revealed that all the sets of connector-lines for each prism belong to a linear complex of lines within a 5-system of screws. Each of the sets of lines is reciprocal to a single screw. Adding additional ties along the diagonals of the skew prism faces can easily form reinforced skew prisms.

Unloaded tensegrity prisms that are stable in the sense that they are configurations of minimum potential energy, but they have instantaneous mobility simply because the connector-lines belong to a linear complex. Reinforced tensegrity prisms where additional ties are inserted are completely stable and shown in Figure 1-3.
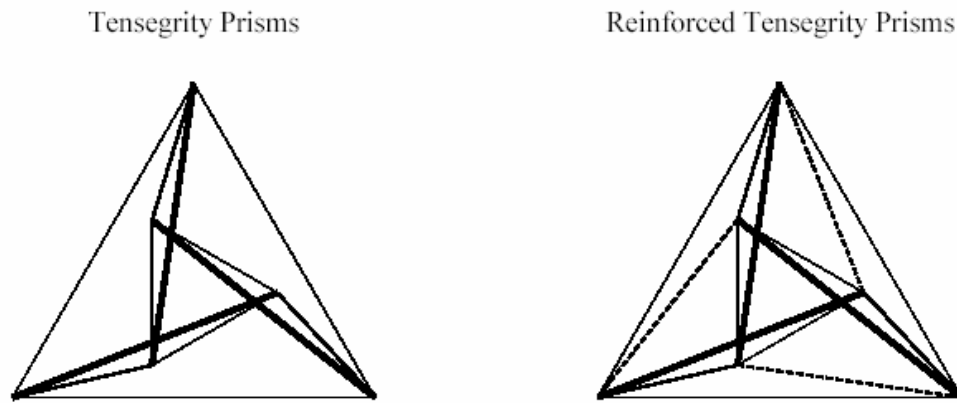
Figure 1-3. Plan view of tensegrity and corresponding reinforced prisms.

Pellegrino [2] shows how deployable structures are able to change their shape from a packed configuration to operational form. Usually, energy is stored in the structure when packed and released during unpacking when the operational configuration is required. Some simple examples are a spring loaded umbrella, a retractable roof of a car, and a space radio telescopic antenna. Deployable structures are used for ease of transportation and storage. The essential requirement is that the transformation process should be autonomous and reliable, and without causing any damage to nearby structures.

Examples of deployable structures that undergo large geometric transformations are coiled rods, flexible shells, lattice column, and membranes. A deployable structure using lattice columns with several longitudinal elements, called longerons, braced at regular intervals by short members perpendicular to the longerons and by diagonal members, has

low dead weight but can support high load. This structure has a very small wind resistance in atmosphere and minimal particle meteoroid damage in space.

Tarnai [3] presents a brief analogy between compatibility and equilibrium of a finite linkage mechanism. Some linkages are known whose degree of freedom in some specific position is greater than the expected degree of freedom as calculated from the equilibrium equation. An example is a four bar linkage with equal length opposite bars. The linkage has two shapes: one associated with a parallelogram and the other with an anti parallelogram. Plotting these angles as a function of each other provides two curves with a common point. This point is when all four bars lie on a straight line. At this instant the mechanism may move from the parallelogram configuration to the anti-parallelogram configuration.

Duffy et al. [4-6] analyze a three dimensional tensegrity structure that is made up of elastic and rigid elements (Figure 1-4). In this assembly, the elastic elements are under tension and the rigid elements under compression. The papers present the static position analysis problem and determine the position assumed by the structure when external loads are applied, and when the system is presented by changing the free lengths of the compliant elements.
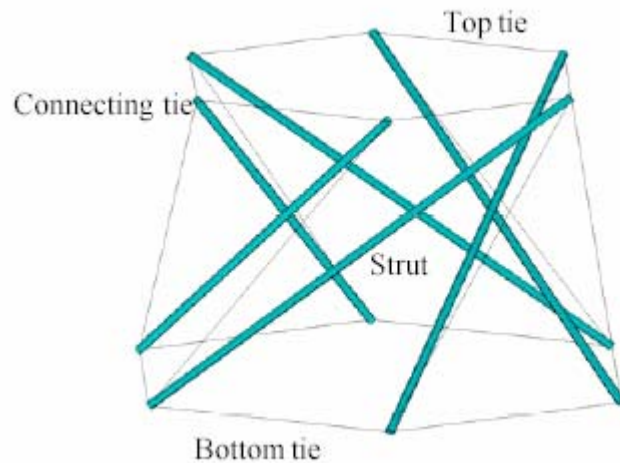
Figure 1-4. Component of a tensegrity structures.

The mathematical formulation to find the equilibrium positions of the structure is based on the virtual work principle. The obtained equations are solved using numerical methods. Some assumptions are made to simplify the derivation of the mathematical model. These assumptions are the absence of internal dissipative forces and the manner by which the external forces are applied. The numerical method determines the coordinates of the strut end points in the equilibrium position. A force balance is then conducted to validate the results.

Stern [7] presents the position analysis of a symmetric n-strut tensegrity system. A three dimensional n-struts system with two platforms, one at the top and the other at the bottom, is considered in the analysis. A static analysis of the internal forces is conducted on the top and bottom platforms. The relationship between the geometry of the structure and the internal forces are investigated. Tensegrity structures with different number of struts are analyzed. The results of each analysis are compared with the results of other analyses to obtain common patterns in all systems by relating the results to the number of

struts in the systems. The patterns are formulated into equations based on the number of struts in the system.

The formulation is done for 3, 4, 5, and 6 struts tensegrity systems with solid top and bottom platforms. For example, a plan view of 4 strut tensegrity system is shown in Figure 1-5.
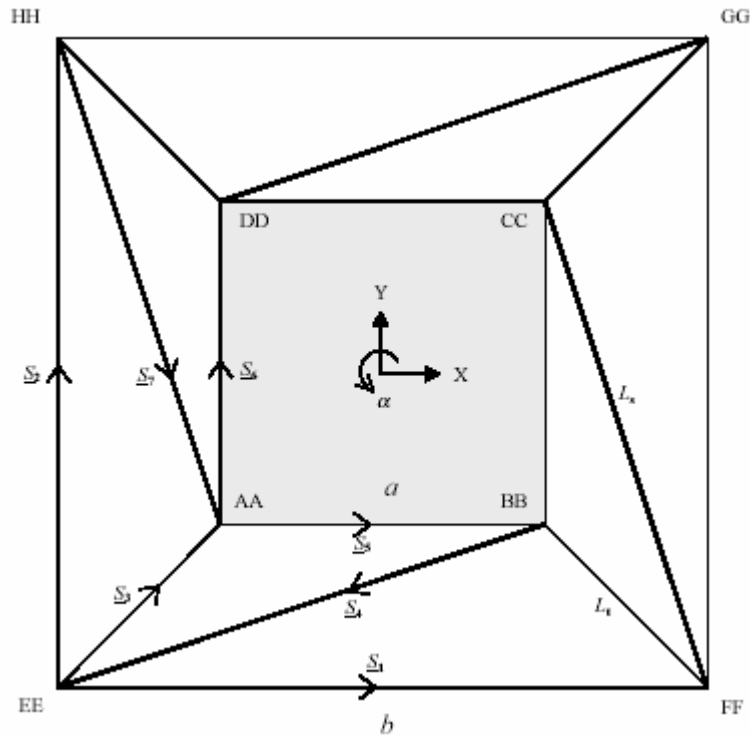
Figure 1-5. 4-strut tensegrity, 3D, top view.

Marshall [8] introduces a parallel platform device that incorporates tensegrity principles. The device, shown in Figure 1-5, replaces the struts, of a tensegrity structure with prismatic actuators, and each elastic member with a cable-spring combination in series. The length of the three prismatic actuators and the length of the cables that are in series with the spring are adjustable and thus the device has six degrees of freedoms. This study shows that in order to achieve an arbitrary position and orientation of the top platform, an external wrench must be applied to maintaining equilibrium. This study also

shows that the device's compliance characteristics can be varied while maintaining its position and orientation.

A reverse analysis of the device is presented in which the desired pose and total potential energy are given and the lengths of the prismatic actuators and the cables are determined. The effect of a seventh leg, another prismatic actuator, is also analyzed and found to satisfactorily implement the needed external wrench (Figure 1-6).

Figure 1-6. Tensegrity platform.

Gantes and Konitopoulou [9] discusses bi-stable deployable structures. A bi-stable structure is self standing and stress free when fully closed or fully deployed. It exhibits incompatibilities between the member lengths at intermediate geometric configurations during the deployment process, which leads to the occurrence of second-order strains and stresses resulting in a snap-through phenomenon that "locks" the structures in their deployed configuration. Until now the geometric shapes that were possible in the deployed configuration were only flat or curved with constant curvature. This limitation is addressed in the paper by proposing a geometric design methodology for deployable arches of arbitrary curvature, accounting also for the discrete joint size, and applying it successfully for the geometric design of a semi-elliptical arch. The arch is then modeled with finite elements, and a geometrically nonlinear analysis is performed in order to

verify the deploy ability feature. Further verification is provided by the construction of a small-scale physical model. A preliminary structural design indicates the overall feasibility of the arch for short to medium spans and light loads.

Deployable structures are prefabricated space frames consisting of straight bars linked together in the factory as a compact bundle, which can then be unfolded into large-span, load bearing structural shapes by simple articulation. Because of this feature they offer significant advantages in comparison to conventional, non-deployable structures for a wide spectrum of applications ranging from temporary structures to the aerospace industry, being mainly characterized by their feature of transforming and adapting to changing needs.

From a structural point of view, deployable structures have to be designed for two completely different loading conditions, under service loads in the deployed configuration, and during deployment. The structural design process is very complicated and requires successive iterations to achieve some balance between desired flexibility during deployment and desired stiffness in the deployed configuration From a geometric point of view, the whole idea of this type of deployable structure is based on the so-called scissor-like elements, pairs of bars connected to each other at an intermediate point through a pivotal connection which allows them to rotate freely about an axis perpendicular to their common plane but restrains all other degrees of freedom, while, at the same time, their end points are hinged to the end points of other scissor-like elements.

Yin et al. [10] present a special planar three-spring mechanism that is designed to control contact forces. An energy function is defined to describe the behavior of this kind of mechanism and it can be used to perform the catastrophe analysis of this mechanism.

The analysis result can be used as a design and control tool. By comparing the three-spring system and a two-spring system, it was found that the three-spring mechanism has better stability than the two-spring system. A three-spring mechanism which can be used to control a general contact force in a plane is also analyzed.

Intuitively, Yin et al. [10] showed a catastrophe occurs whenever a smooth change of parameters gives rise to a discontinuous change in behavior. A well known example that can easily be made to demonstrate a catastrophe is Zeeman's catastrophe machine (see Figure1-7). Zeeman's machine can be constructed by attaching two linear springs to a single point C on a disk that can rotate about 0. One of the springs is attached to a fixed pivot at point A and the other spring is attached to point B which can be moved in the x, y plane. The position of point B is called the controlling parameter as it dictates the position of the disk which is defined by the angle between OA and OM.
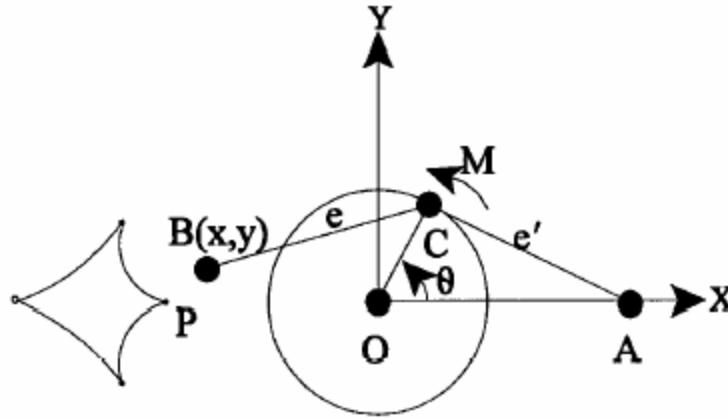
Figure 1-7. Zeeman's catastrophe machine.

Parallel compliant mechanisms are classically known for their role in mounting and suspension systems. Recently, however, they have been instrumental in the field of robotic force control. A new theory for the simultaneous control of displacement and force for a partially constrained end-effecter has been proposed in Yin et al. [10].

Figure 1-8 shows a planar compliant mechanism which is actually a planar two-spring system. It consists of a pair of linear springs connected at one end to a movable base and at the other end to a common pivot which is the axis of the wheel contacting to a surface. The base is connected to a planar two freedom P-P (P denotes prismatic pair) manipulator . The contact force can be controlled by displacing the two prismatic joints of the manipulator. The required displacements can be calculated from the stiffness mapping. This kind of control was called kinestatic control by Griffis and Duffy [11]. In order to design the planar two-spring system it is necessary to compute a spring stiffness which will generate a range of displacements of the movable base which can be produced by the prismatic joints over a required range of change in contact force. Clearly if the system is over-designed and the spring stiffness is very high it is always possible to generate any necessary changes in contact force. However such a system will be too sensitive to errors because very small displacements of the platform will generate large changes in contact force. On the other hand if the springs are too soft there can be stability problems.
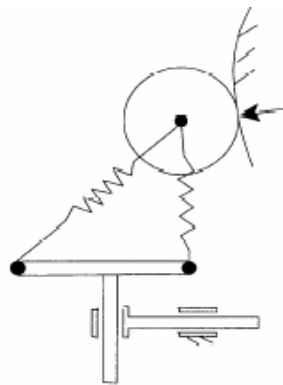
Figure 1-8. Planar two-spring system.

Yin et al. [10] are also considered another compliant mechanism. That is shown in Figure 1-9. It consists of three linear springs joined to the triangular frame fixed points

and connected at the axis of the wheel. The triangular frame is connected to the planar two freedom P-P manipulator. This mechanism is a special three-spring system. The catastrophe analysis of this system was presented. Comparing the results demonstrates that three-spring system has better stability characteristics than the two-spring system.
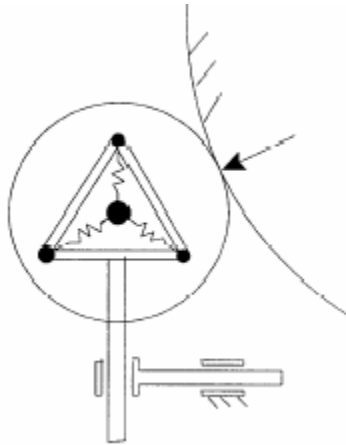
Figure 1-9. Special three-spring system.

A more general planar three-spring compliant mechanism is shown in Figure 1-10. This mechanism is connected to the planar three freedom R-R-R (R denotes revolute pair) manipulator. It can be used to control both force and moment. The catastrophe analysis of this mechanism was performed in Yin et al. [10].
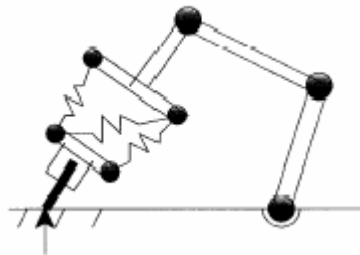
Figure 1-10. General planar three-spring compliant mechanism.

Crane and Duffy [12] studied kinematics of robot manipulators and Crane and Bayat [13] presented cases of two dimensional tensegrity structures that proved to be mathematically challenging in spite of their simple configurations. Different methods

were tried to solve these problems. It was expected the final closed form solution be a simple mathematical expression. In each case the intermediate mathematical expressions become complicated to solve and the relevant matrices become large to manipulate.

Roth and Whiteley [14] propose a technology based on tensegrity for tough, rigid, large-scale domes that are also economical to construct. The development of a structural technology to economically cover large areas would be useful for warehouses, permanent or temporary protection for archaeological and other vulnerable sites, large-scale electrical or electromagnetic shielding and exclusion or containment of flying animals or other objects. Structures based on such a technology can serve as frameworks in which environmental control, energy transformation and food production facilities could be embedded. The space application is also possible by using self-deployed structures. Summary Advantages are improved rigidity, ethereal, resilient, equal-length struts, simple Joints.

The resulting list represents an initial attempt to identify applications for which the technology would be suitable. As the technology develops and is tested against them, some or all of the applications may be winnowed out and other suitable applications not in the list may become apparent. The current list is as follows:

Superstructures for embedded substructures allow the substructures to escape terrestrial confines where this is useful (e.g. in congested or dangerous areas, urban areas, flood plains or irregular, delicate or rugged terrains).

1.   Economic large-scale protection of storage, archaeological, agricultural, construction or other sites.

2.   Refugee or hiking shelters.

3.   Frames over cities for environmental control, energy transformation and food production.

4.   Large-scale electrical or electromagnetic shielding.

5.   Exclusion or containment of flying animals or other objects.

6.   Spherical superstructures for space stations.

7.   Earthquake-resistant applications.

These structures are extremely resilient and testing would very likely show they could withstand large structural shocks like earthquakes. Thus, they would likely be desirable in areas where earthquakes are a problem. A tensegrities, pneumatic, structure performed very well during recent earthquakes in Japan.

- Low-environmental-impact shells for musical performances.

- Indoor/outdoor pavilions for trade shows etc.

- Supports to hold sunscreen protection for vulnerable amphibians.

- Watersheds to keep rain water from percolating through contaminated soils into groundwater, perhaps temporarily during in-situ remediation.

- Frames for hanging plants or other objects to dry.

- Pergola, trellis, or topiary framework.

- Micro-meteorite protection, sun-shielding for Martian colonies.
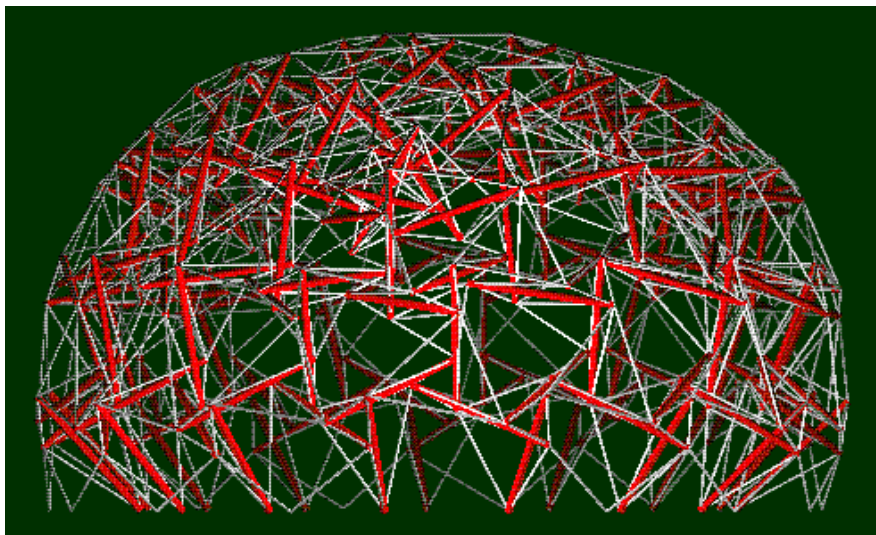
Figure 1-11. A representation of a dome which utilitizes tensegrity solutions' technology

Skelton et al. [15] present a solution so the tensegrity structures are reduced to linear algebra problems, after first characterizing the problem in a vector space where direction cosines are not needed. That is, we describe the components of all members by vectors as opposed to the usual practice of characterizing it as static problem in terms of the magnitude of tension and compression. While this approach transfer the problem into vector space to describe the problem, the advantage is that the vector space makes the mathematical structure of the problem amenable to linear algebra treatment for both two and three dimensional tensegrity structures.

This paper characterizes the static equilibrium some tensegrity structures. Furthermore, it uses vectors to describe each element (springs and studs), eliminating the need to use direction cosines and the subsequent transcendental functions that followthe use of trigonometry equations.

In this paper, the authors choose to represent a tensegrity structure as an oriented graph in real three dimensional space R3 defined in terms of nodes and directed branches which are all represented as vectors in R3. A loop is any closed path in the graph.

The advantage of this approach is that the both the magnitude and the direction cosines of the forces are contained in vectors which can be solved using linear algebra. Thus linear algebra plays a larger role in this approach compared to the usual approach in mechanics and finite element methods using direction cosines.

In this oriented graph, the nodes consist of the ends of the bars. Hence if there are n bars, then there are 2n nodes. There are two types of directed branches; the string branch (in vectors) and the bar branches (in vectors).

Geometric connectivity of the structure maintained such that each directed branch can undergo a displacement in reaching its equilibrium state. String vectors can change both their length and orientation while bar vectors can only change their orientation. Node vectors can change both their length and orientation but subject to a Law of Geometric Connectivity is stated as follows: The vector sum of all branch vectors in any loop is zero. These loop equations are in the form of a set of linear algebraic equations in the branch vectors.

In this study of tensegrity structures, force equilibrium are such that spring can only take tension and bars sustain compressive forces. We therefore choose to distinguish between the string (or tensile) forces and the bar (or compressive) forces which are defined in terms of the string and bar vectors respectively.

This paper reduces the study of the tensegrity equilibrium to a series of linear algebra problems using directed graph theory. Of course the existence conditions for the linear algebra problems are nonlinear in the design variables. The presented procedure and formulation give some insight to solution of tensegrity structures and identifies the free parameters that may be used to achieve desired structural shapes.

Tibert [16] presents adjustable tensegrity structure. A tensegrity structure is a lightweight consisting of compression members surrounded by a network of tension members. They can be easily dismantled providing possibilities for reusable and modular structures. Tensegrities adapt their shape by self adjusting their tension and compression in their members. They can adapt to changing environment when they are equipped with sensors and actuators.

A full-scale prototype of an adjustable tensegrity is built and tested at Swiss Federal Institute of Technology. This paper begins with a description of important aspects of the design, assembly, and static testing. Tests show that the structure behaves linearly when subjected to vertical loads applied to a single joint. Nonlinearities are detected for small displacements, for loads applied to several joints and for adjusting combinations of telescoping compression members. To predict behavior, dynamic relaxation—a nonlinear method—has been found to be reliable. Appropriate strut adjustments found by a stochastic search algorithm are identified for the control goal of constant roof slope and for the load conditions studied. When adjusting struts, an excessive number of adjustable members does not necessarily lead to improved performance.

Tensegrity is an abbreviation of tensile–integrity as termed by one R. Buckminster Fuller. He described tensigrity as ''small islands of compression in a sea of tension.'' This is a description of a network of light tension members that provide rigidity to a limited number of discontinuous compression members. Some researchers have found that discontinuous compressions are not a necessity for creating a tensegrity and more efficient structures can be built if compression elements are allowed to join. Defining a tensegrity structure as any self-stressing structure composed of struts and cables would include structures such as a bicycle wheel. In order to add precision terminology, a definition has been proposed by Motro and Raducanu. They describe a tensegrity as a stable system that contains a discontinuous set of components in compression inside a network of components in tension.

Several independent efforts have contributed to the invention and development of tensegrity structures. Le Ricolais designed many unusual lightweight cable–strut structures. The sculptor Kenneth Snelson started in 1948 with sculptures employing members in continuous tension and discontinuous compression. Applications of the tensegrity principle in nature have recently been found through studies that use a tensegrity model to describe observations of how cells respond to stress.

Transforming tensegrity from sculptures into practical structural has been a challenge that began with several studies of their geometric, nonlinear behavior and states of self stress. Reasons for the lack of test data are related to the fabrication assembly process as well as to the lack of rigidity unless pre-stressed members are used.

Joint design is the biggest challenge in constructing a full-scale tensegrity. Although it is the problem for all space structures, tensegrities present particular challenges. Joints need to be pin jointed, modular, and light in order to take advantage of tensegrities ease of dismantling and potential reuse.

In addition to the complexities of constructing the structure itself, simulation of the behavior is not straightforward. Tensegrity structures exhibit geometric nonlinear behavior. In addition, as it is common with other pin-jointed structures, nodal friction should be included in the simulation of a truss structure for precision control. Finally, for full-scale construction, assembly sequences, fabrication tolerances, and construction lack-of-fit need to be considered.

Fest et al. [17] present a study of adjustable tensegrity structures. A tensegrity structure is consisting of compression members surrounded by a network of tension members. A tensegrity structure is a lightweight structure. Tensegrity structures can be

easily dismantled providing possibilities for reusable and modular structures. Tensegrities adapt their shape by self adjusting their tension and compression in their members. They can adapt to changing environment when they are equipped with sensors and actuators.

A full-scale prototype of an adjustable tensegrity is built and tested at Swiss Federal Institute of Technology. This paper presents important aspects of the design, assembly, and static testing. Tests show that the structure behaves linearly when it is subjected to vertical loads applied to a single joint. Nonlinearities are observed when loads applied to several joints and for adjusting combinations of telescoping compression members. Appropriate strut adjustments found by a search algorithm for the load conditions studied. When adjusting struts, an excessive number of adjustable members do not necessarily lead to equilibrium.

Active tensegrity structures have the potential to widen the scope for innovative, lightweight, and reusable structural systems. Lessons learn through the construction and testing of an adjustable full-scale prototype can be used in construction and design of practical tensegrity structures. This work expected to contribute to the development of tensigrity and improve in their design and performance during service.

CHAPTER 2
TWO SPRING PLANAR TENSEGRITY SYSTEM

2.1 Introduction

This Chapter presents two approaches to solve the forward position analysis of a

two-strut tensegrity system with two compliant ties and two non-compliant ties (Figure 2-

1).
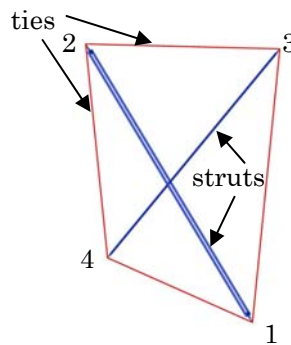
Figure 2-1. Two dimensional tensegrity structure.

Here it is assumed that the lengths of the struts and noncompliant ties are known

together with the spring constants and free lengths of the two compliant ties. The

objective is to determine all possible equilibrium poses for the device when no external

loads are applied. Gravity loads are neglected.

The objective of this effort is to determine, in closed-form, all possible equilibrium

deployed positions in stable condition (minimum potential energy) of a planar tensegrity

system wherein two of the ties are compliant. Figure 2-2 shows the system which is

comprised of two struts (compression members a_{12} and a_{34}), two non-compliant ties

(tension members a_{41} and $a_{23)}$, and two elastic tensile members (springs), one connected

between points 1 and 3 and one between points 2 and 4. It should be noted in Figure 2-2

that strut a_{34} passes through a slit cut in strut a_{12} and as such the two struts do not

intersect or collide.
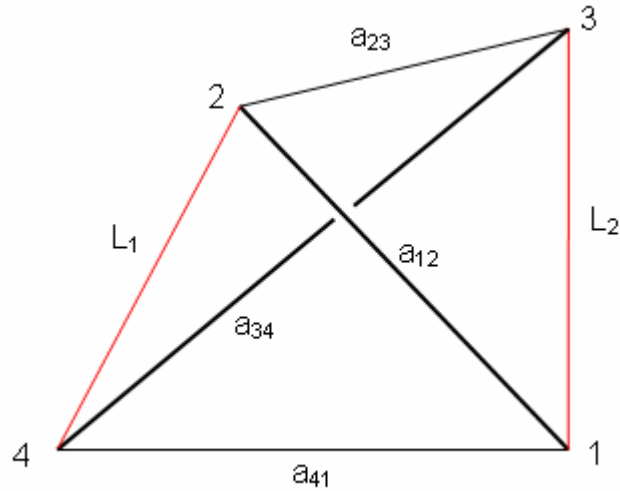
Figure 2-2. Two-spring planar tensegrity system.

2.2 Problem Statement

The problem statement for the two-spring plane tensegrity system can be explicitly

written as

given:

a_{12}, a_{34} lengths of struts

a_{23}, a_{41} lengths of non-compliant ties

k_1, L_{01} spring constant and free length of compliant tie between points 4 and 2

k_2, L_{02} spring constant and free length of compliant tie between points 3 and 1

find:

L_1 length of spring 1 at equilibrium position

L_2 length of spring 2 at equilibrium position

It should be noted that the problem statement could be formulated in a variety of

ways, that is, a different variable (such as the relative angle between strut a_{34} and tie a_{41})

could have been selected as the generalized parameter for this problem. Two solution

approaches are presented.

2.3 Approach 1: Determine (L_1, L_2) to Minimize Potential Energy

2.3.1 Development of Geometric Equations

Figure 2-3 shows the three angles θ_4, θ'_4, and θ''_4 which must satisfy the relation

$$\theta_4 + \theta_4' = \pi + \theta_4" \ .$$

(2-1)

Figure 2-4 shows the triangle formed by side a_{34}, a_{23}, and L_1. A cosine law for this

triangle can be written as

$$\frac{L_1^2}{2} + \frac{a_{34}^2}{2} + L_1\, a_{34} \cos\theta_4' = \frac{a_{23}^2}{2} \ .$$

(2-2)
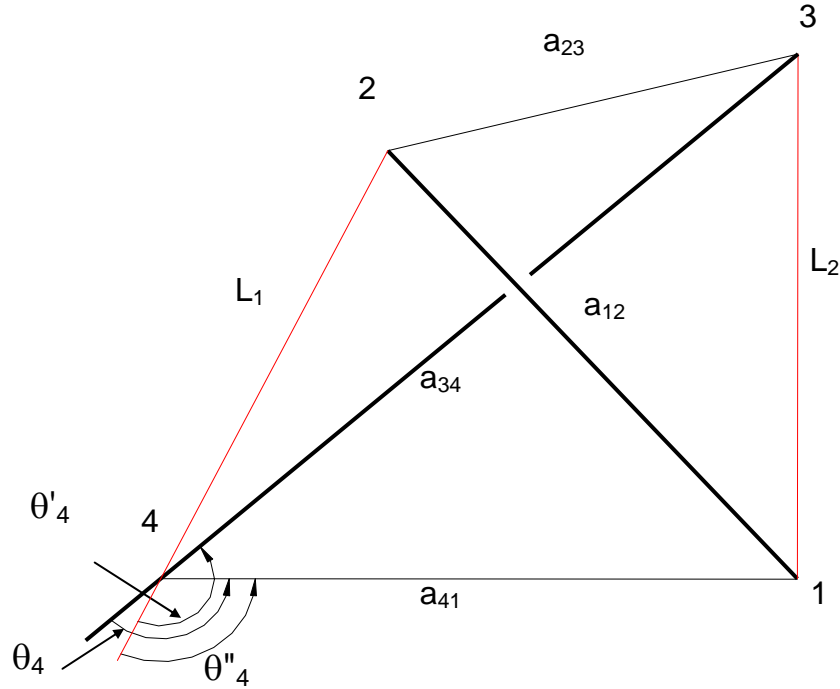
Figure 2-3. Identification of angles θ_4, θ'_4, and θ''_4.
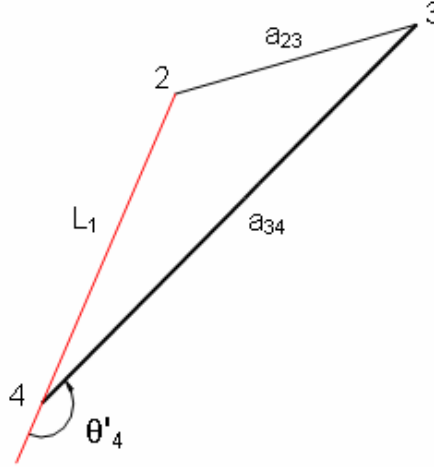
Figure 2-4. Triangle 4-1-3.

Solving for $\cos\theta_4'$ yields

$$\cos\theta_4' = \frac{a_{23}{}^2 - L_1{}^2 - a_{34}{}^2}{2 L_1 a_{34}}.$$

(2-3)

Figure 2-5 shows the triangle formed with sides a_{41}, a_{12}, and L_1. A cosine law for this triangle can be written as

$$\frac{L_1{}^2}{2} + \frac{a_{41}{}^2}{2} + L_1 a_{41} \cos\theta_4'' = \frac{a_{12}{}^2}{2}.$$

(2-4)

Solving for $\cos\theta_4''$ yields

$$\cos\theta_4'' = \frac{a_{12}{}^2 - L_1{}^2 - a_{41}{}^2}{2 L_1 a_{41}}.$$

(2-5)

Figure 2-6 shows the triangle formed by sides a_{41}, a_{34} and L_2. A cosine law for this triangle can be written as

$$\frac{a_{34}{}^2}{2} + \frac{a_{41}{}^2}{2} + a_{34} a_{41} \cos\theta_4 = \frac{L_2{}^2}{2}$$

(2-6)

Solving for $\cos\theta_4$ yields

$$\cos\theta_4 = \frac{L_2{}^2 - a_{34}{}^2 - a_{41}{}^2}{2 a_{34} a_{41}}$$

(2-7)
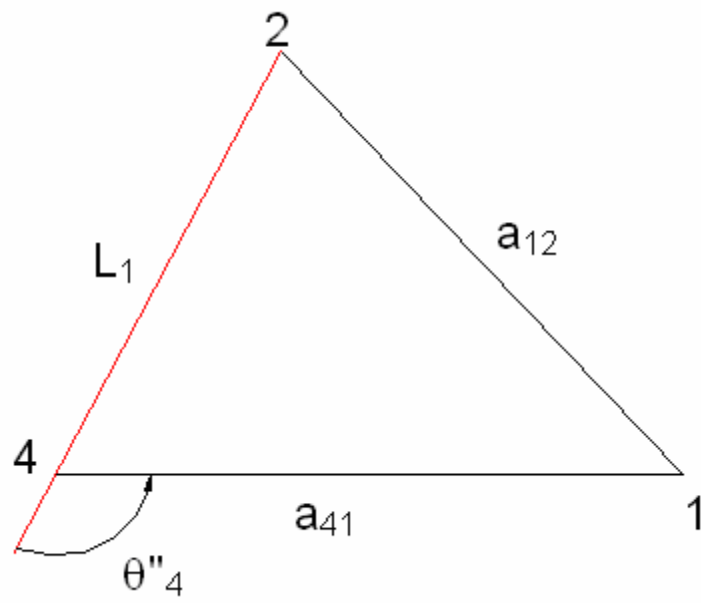
Figure 2-5. Triangles 4-1-2.
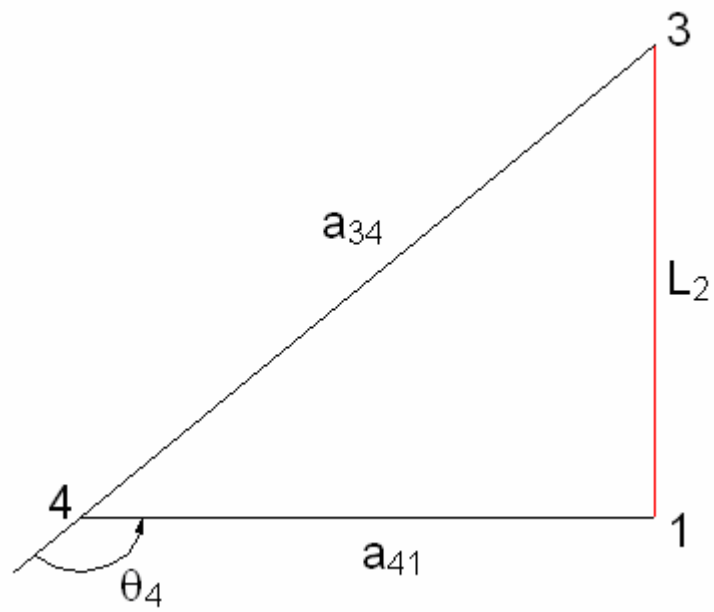
Figure 2-6. Triangles 4-1-3.

Equating the cosine of the left and right sides of 2-1 yields

$$\cos(\theta_4 + \theta_4') = \cos(\pi + \theta_4'') \quad . \tag{2-8}$$

Expanding this equation yields

$$\cos\theta_4 \cos\theta_4' - \sin\theta_4 \sin\theta_4' = -\cos\theta_4'' \quad . \tag{2-9}$$

Rearranging 2-9 yields

$$\cos\theta_4 \cos\theta_4' + \cos\theta_4'' = \sin\theta_4 \sin\theta_4' \tag{2-10}$$

Squaring both sides of 2-10 gives

$$(\cos\theta_4)^2 (\cos\theta_4')^2 + 2 \cos\theta_4 \cos\theta_4' \cos\theta_4'' + (\cos\theta_4'')^2 = (\sin\theta_4)^2 (\sin\theta_4')^2 \quad . \tag{2-11}$$

Substituting for $(\sin\theta_4)^2$ and $(\sin\theta_4')^2$ in terms of $\cos\theta_4$ and $\cos\theta_4'$ gives

$$(\cos\theta_4)^2 (\cos\theta_4')^2 + 2 \cos\theta_4 \cos\theta_4' \cos\theta_4'' + (\cos\theta_4'')^2 =$$

$$(1-\cos^2\theta_4)(1-\cos^2\theta_4') \quad . \tag{2-12}$$

Equations 2-3, 2-5, and 2-7 are substituted into 2-12 to yield a single equation in the parameters L_1 and L_2 which can be written as

$$A L_2^4 + B L_2^2 + C = 0 \tag{2-13}$$

where

$$A = L_1^2, \quad B = L_1^4 + B_2 L_1^2 + B_0, \quad C = C_2 L_1^2 + C_0 \tag{2-14}$$

and

$$B_2 = -(a_{23}^2 + a_{34}^2 + a_{41}^2 + a_{12}^2),$$

$$B_0 = (a_{12} - a_{41})(a_{12} + a_{41})(a_{23} - a_{34})(a_{23} + a_{34}) ,$$

$$C_2 = (a_{34} - a_{41})(a_{34} + a_{41})(a_{23} - a_{12})(a_{23} + a_{12}) ,$$

$$C_0 = (a_{41}a_{23} + a_{34}a_{12})(a_{41}a_{23} - a_{34}a_{12})(a_{41}^2 + a_{23}^2 - a_{12}^2 - a_{34}^2) \tag{2-15}$$

Equation 2-13 expresses length L_2 as a function of the length L_1. The function is quadratic with respect to $(L_2)^2$ and $(L_1)^2$ and thus there will be two possible values for $(L_2)^2$ for each value of $(L_1)^2$.

As a verification of Equation 2-13 a numerical example is presented to show how four values of L_2 can occur for a given value of L_1. For this example, assume $a_{41} = 1.0$ m, $a_{12} = 2.0$ m, $a_{23} = 2.0$ m, and $a_{34} = 2.5$ m. Further, assume the value of L_1 is given as $L_1 = 2.25$ m. Evaluating the coefficients in Equations 2-15 and 2-14 and substituting into Equation 2-13 gives

$$(5.0625)\, L_2^4 + (-58.3242)\, L_2^2 + (110.2500) = 0 \,. \tag{2-16}$$

Solving this equation for L_2 yields four answers;

$$L_{2a} = -3.0228, \; L_{2b} = 3.0228, \; L_{2c} = -1.5438, \; L_{2d} = 1.5438 \,. \tag{2-17}$$

Figure 2-7 shows the mechanism in four configurations where for each of these configurations L_2 can have a positive or negative value, thereby giving eight possible states of the mechanism. However, due to the symmetry associated with the reflected solutions about a_{41}, only four values of L_2 (or two values of L_2^2) exist. This example was presented to show that the degree of the polynomial which relates L_1 and L_2 is indeed fourth order.
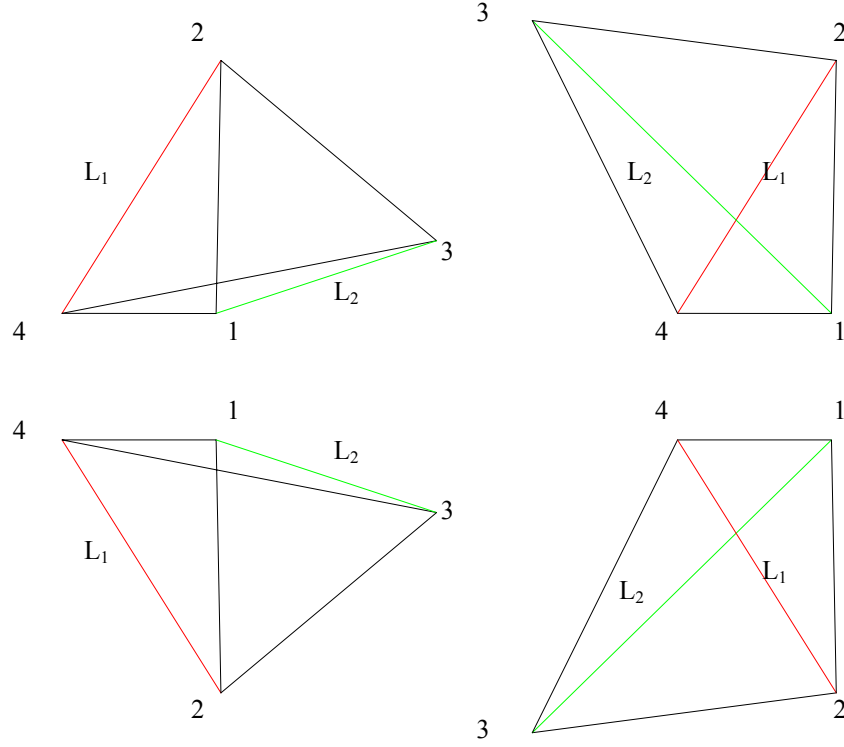
Figure 2-7. Possible configurations for numerical example.

The potential energy of the system can be evaluated as

$$U = \frac{1}{2}k_1 (L_1 - L_{01})^2 + \frac{1}{2}k_2 (L_2 - L_{02})^2 \quad .$$

(2-18)

At equilibrium, the potential energy will be a minimum. This condition can be

determined as the configuration of the mechanism whereby the derivative of the potential

energy taken with respect to the length L_1 equals zero, i.e.

$$\frac{dU}{dL_1} = k_1 (L_1 - L_{01}) + k_2 (L_2 - L_{02})\frac{dL_2}{dL_1} = 0 \quad .$$

(2-19)

The derivative dL_2/dL_1 can be determined via implicit differentiation from

Equation 2-13 as

$$\frac{dL_2}{dL_1} = \frac{-L_1[L_2^{\,2}(L_2^{\,2} + 2L_1^{\,2} - a_{23}^{\,2} - a_{41}^{\,2} - a_{34}^{\,2} - a_{12}^{\,2}) + (a_{12}^{\,2} - a_{23}^{\,2})(a_{41}^{\,2} - a_{34}^{\,2})]}{L_2[L_1^{\,2}(L_1^{\,2} + 2L_2^{\,2} - a_{23}^{\,2} - a_{41}^{\,2} - a_{34}^{\,2} - a_{12}^{\,2}) + (a_{12}^{\,2} - a_{41}^{\,2})(a_{23}^{\,2} - a_{34}^{\,2}))]} \quad .$$

(2-20)

Substituting Equation 2-20 into Equation 2-19 and regrouping gives

$$D\,L_2^5 + E\,L_2^4 + F\,L_2^3 + G\,L_2^2 + H\,L_2 + J = 0 \qquad (2\text{-}21)$$

where

$$D = D_1\,L_1, \;\; E = E_1\,L_1, \;\; F = F_3\,L_1^3 + F_2\,L_1^2 + F_1\,L_1, \;\; G = G_3\,L_1^3 + G_1\,L_1,$$

$$H = H_5\,L_1^5 + H_4\,L_1^4 + H_3\,L_1^3 + H_2\,L_1^2 + H_1\,L_1 + H_0, \;\; J = J_1\,L_1 \qquad (2\text{-}22)$$

and where,

$$D_1 = k_2, \;\; E_1 = -k_2\,L_{02},$$

$$F_3 = 2\,(k_2 - k_1), \; F_2 = 2\,k_1\,L_{01}, \; F_1 = -\,k_2\,(a_{12}^2 + a_{23}^2 + a_{34}^2 + a_{41}^2),$$

$$G_3 = -2\,k_2\,L_{02}, \; G_1 = k_2\,L_{02}\,(a_{12}^2 + a_{23}^2 + a_{34}^2 + a_{41}^2),$$

$$H_5 = -\,k_1, \; H_4 = k_1\,L_{01}, \; H_3 = k_1\,(a_{12}^2 + a_{23}^2 + a_{34}^2 + a_{41}^2),$$

$$H_2 = -\,k_1\,L_{01}\,(a_{12}^2 + a_{23}^2 + a_{34}^2 + a_{41}^2),$$

$$H_1 = -\,k1\,(a_{34}^2 - a_{23}^2)\,(a_{41}^2 - a_{12}^2) + k_2\,(a_{34}^2 - a_{41}^2)\,(a_{23}^2 - a_{12}^2),$$

$$H_0 = k_1\,L_{01}\,(a_{34}^2 - a_{23}^2)\,(a_{41}^2 - a_{12}^2),$$

$$J_1 = k_2\,L_{02}\,(a_{34}^2 - a_{41}^2)\,(a_{12}^2 - a_{23}^2) \qquad (2\text{-}23)$$

Equations 2-13 and 2-21 represent two equations in the two variables L_1 and L_2. The simultaneous solution of these equations is a necessary condition that the system is in equilibrium.

2.3.2 Sylvester's Solution Method

A system of *m* equations of order *n* in the variable Y may be written as (Gantes and Konitopoulou [9])

$$A_{1,n}\,Y^n + A_{1,n\text{-}1}\,Y^{n-1} + A_{1,n\text{-}2}\,Y^{n-2} + \ldots + A_{1,0} = 0$$

$$A_{2,n}\,Y^n + A_{2,n\text{-}1}\,Y^{n-1} + A_{2,n\text{-}2}\,Y^{n-2} + \ldots + A_{2,0} = 0 \,,$$

$$A_{m,n}\,Y^n + A_{m,n\text{-}1}\,Y^{n-1} + A_{m,n\text{-}2}Y^{n-2} + \ldots + A_{m,0} = 0 \,. \qquad (2\text{-}24)$$

The terms $U_n = Y^n$, $U_{n-1} = Y^{n-1}$, ..., $U_1 = Y$, $U_0 = 1$ may be substituted into Equation 2-24 to yield

$$A_{1,n} U_n + A_{1,n-1} U_{n-1} + A_{1,n-2} U_{n-2} + ... + A_{1,0} U_0 = 0$$

$$A_{2,n} U_n + A_{2,n-1} U_{n-1} + A_{2,n-2} U_{n-2} + ... + A_{2,0} U_0 = 0 ,$$

$$A_{m,n} U_n + A_{m,n-1} U_{n-1} + A_{m,n-2} U_{n-2} + ... + A_{m,0} U_0 = 0 .$$
(2-25)

Equation 2-25 represents a system of m homogeneous equations with n+1 unknowns (the term U_0 will be treated as an unknown for the time being) and can be written in matrix form as

$$\begin{bmatrix} A_{1,n} & A_{1,n-1} & \cdots & A_{1,0} \\ A_{2,n} & A_{2,n-1} & \cdots & A_{2,0} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m,n} & A_{m,n-1} & \cdots & A_{m,0} \end{bmatrix} \begin{bmatrix} U_n \\ U_{n-1} \\ \cdots \\ U_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix} .$$
(2-26)

If the number of equations is less than the number of unknowns, additional equations can be created by multiplying selected equations from set of Equation 2-24 by powers of Y to obtain a new system. For example, multiplying the first equation from the set of Equation 2-24 by Y will introduce one new equation and one new unknown, Y^{n+1} (which will be represented by U_{n+1}). Now, multiplying the second equation of the set of Equation 2-24 by Y will yield another equation, but no new variables are introduced. Ultimately, the system will be formed where the new total number of homogeneous equations equals the new total number of unknowns. In other words, the coefficient matrix will be a square matrix. Now, one solution to the set of 'homogeneous' equations is the trivial solution where all the U_i terms equal zero. However, this cannot occur since $U_0=1$. Thus it must be the case that the set of equations are linearly dependent which means that the determinant of the coefficient matrix must equal zero.

Now consider that the terms $A_{i,j}$ of the coefficient matrix are polynomial functions

of order q of another variable called X, that is

$$A_i = a_i \, X^q + b_i \, X^{q-1} + c_i \, X^{q-2} + \ldots + z_i = 0 \qquad (2\text{-}27)$$

where a_i, b_i, c_i, $\ldots$ z_i are constants. Evaluation of the determinant of the

coefficient matrix will result in a polynomial in the variable X that must equal zero so

that the set of equations in the variables U_i will be linearly dependent. This solution

approach will now be applied to Equations 2-13 and 2-21 which represent two equations

in the two unknowns L_1 and L_2.

2.3.3 Solution of Geometry and Energy Equations

Equations 2-13 and 2-21 can be solved by using Sylvester's variable elimination

procedure by multiplying Equation 2-13 by L_2, L_2^2, L_2^3, and L_2^4 and Equation 2-21 by L_2,

L_2^2, L_2^3 to yield a total of nine equations that can be written in matrix form as

$$\begin{bmatrix} 0 & 0 & 0 & D & E & F & G & H & J \\ 0 & 0 & 0 & 0 & A & 0 & B & 0 & C \\ 0 & 0 & 0 & A & 0 & B & 0 & C & 0 \\ 0 & 0 & D & E & F & G & H & J & 0 \\ 0 & 0 & A & 0 & B & 0 & C & 0 & 0 \\ 0 & D & E & F & G & H & J & 0 & 0 \\ 0 & A & 0 & B & 0 & C & 0 & 0 & 0 \\ D & E & F & G & H & J & 0 & 0 & 0 \\ A & 0 & B & 0 & C & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} L_2^{8} \\ L_2^{7} \\ L_2^{6} \\ L_2^{5} \\ L_2^{4} \\ L_2^{3} \\ L_2^{2} \\ L_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (2\text{-}28)$$

This set of equations can only be solved if the determinant of the 9×9 coefficient

matrix is equal to zero. Expansion of this determinant yields a 30th degree polynomial in

the variable L_1. When the determinant was expanded symbolically, it was seen that the

two lowest order coefficients were identically zero. Thus the polynomial can be divided

throughout by $L_1{}^2$ to yield a 28th degree polynomial. The coefficients of the 28th degree

polynomial were obtained symbolically in terms of the given quantities. They are not

presented here due to their length and complexity.

Values for L_2 that correspond to each value of L_1 can be determined by first solving

Equation 2-13 for four possible values of L_2. Only one of these four values also satisfies

Equation 2-21.

2.3.4 Numerical Examples

Example 1: The following parameters were selected for this numerical example:

Strut lengths	$a_{12} = 3$ in.	$a_{34} = 3.5$ in.
Non-compliant tie lengths	$a_{41} = 4$ in.	$a_{23} = 2$ in.
Spring 1 free length & spring constant	$L_{01} = 0.5$ in.	$k_1 = 4$ lbf/in.
Spring 2 free length & spring constant	$L_{02} = 1$ in.	$k_2 = 2.5$ lbf/in.

Eight real and twenty complex roots were obtained for the 28th degree polynomial

in L_1. Real values for L_1 and the corresponding values of L_2 are shown in Table 2-1.

Table 2-1. Eight real solutions.

Case	1	2	3	4	5	6	7	8
L1, in.	-5.485	-5.322	-1.741	-1.576	1.628	1.863	5.129	5.476
L2, in.	2.333	-2.901	-1.495	1.870	1.709	-1.354	-3.288	2.394

The values of L_1 and L_2 listed in Table 2-1 satisfy the geometric constraints defined

by Equation 2-13 and the energy condition defined by Equation 2-21. Each of these eight

cases was analyzed to determine whether it represented a minimum or maximum

potential energy condition and cases 3, 4, 5, and 6 were found to be minimum states. A

free body analysis of struts a_{12} and a_{34} was performed to show that these bodies were

indeed in equilibrium for each of these four cases. Figure 2-8 shows the four static

equilibrium configurations of the system. Note that there are several cases where the
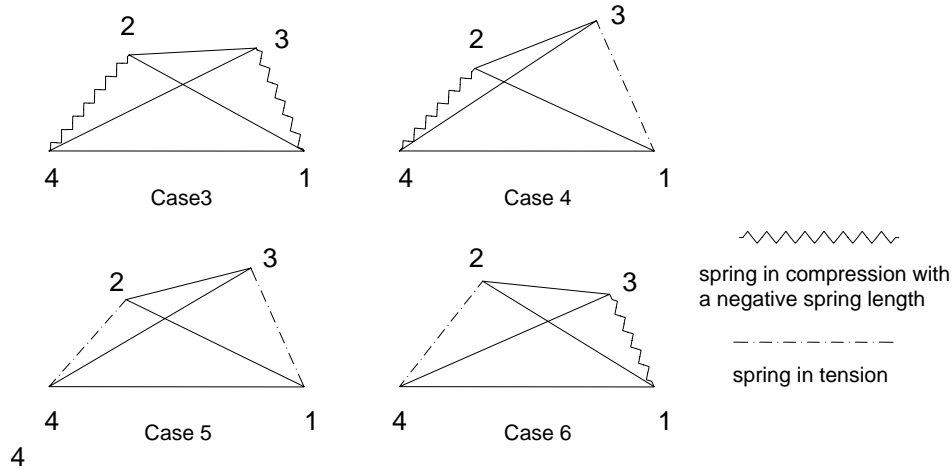
Figure 2-8. Four equilibrium configurations.

spring lengths are negative, such as for example the spring between points 2 and 4 in case

4. In this case the spring is acting to pull point 2 towards point 4 which is counter

intuitive for the normal spring that is in compression.

There are twenty complex roots for L_1 in this example. Unique corresponding

values for L_2 were obtained for each case (Table 2-2). In fact there are corresponding

values for L_2 such that the complex pair of L_1 and L_2 satisfies both Equations 2-13 and

2-21. This result means that no extraneous terms were introduced in the elimination

procedure. These complex roots were not analyzed further.

Table 2-2. Complex solutions.

Case	L_1, inches	L_2, inches
1	-7.0285 – 0.0648i	-0.3225 + 2.5486i
2	-7.0285 + 0.0648i	-0.3224 – 2.5486i
3	-2.0919 – 2.4639i	-6.5995 + 0.3617i
4	-2.0919 + 2.4639i	-6.5995 – 0.3617i
5	-1.7904 – 2.0044i	6.5232 + 0.0738i
6	-1.7904 + 2.0044i	6.5232 – 0.0738i
7	-0.7211 – 0.1827i	0.2411 – 1.2909i
8	-0.7211 + 0.1827i	0.2411 + 1.2909i
9	0.005823 – 1.7651i	-0.05803 – 0.01682i
10	0.005823 + 1.7651i	-0.05803 + 0.01682i
11	0.3062 – 0.4101i	-0.1220 – 0.9493i
12	0.3062 + 0.4101i	-0.1220 + 0.9493i
13	0.8140 – 0.5553i	0.5055 +0.9541i

Table 2-2. Continued.

Case	L_1, inches	L_2, inches
14	0.8140 + 0.5553i	0.5055 +0.9541i
15	2.3264 – 1.9808i	6.1894 + 0.2416i
16	2.3264 + 1.9808i	6.1894 – 0.2416i
17	2.9910 – 2.5413i	-6.2069 – 0.9310i
18	2.9910 + 2.5413i	-6.2069 + 0.9310i
19	7.0351 – 0.1067i	-0.4518 – 2.6442i
20	7.0351 + 0.1067i	-0.4518 + 2.6442i

Example 2: For this case, a C language program was written that would randomly select values for the parameters a_{12}, a_{23}, a_{34}, a_{41}, k_1, L_{01}, k_2, and L_{02} in attempt to find a case where the number of real roots for L_1 was twenty eight. Such a case was not found, but a set of inputs resulting in twenty four real values for L_1 is presented here.

The input parameters for this case were selected as:

Strut lengths $\qquad$ $a_{12} = 3.309848$ in. $\quad a_{34} = 3.002692$ in.

Non-compliant tie lengths $\qquad$ $a_{41} = 7.335484$ in. $\quad a_{23} = 7.978210$ in.

Spring 1 free length & spring constant $\;L_{01} = 0.953703$ in. $\;\; k_1 = 0.193487$ lbf/in.

Spring 2 free length & spring constant $\;L_{02} = 0.607318$ in. $\;\; k_2 = 9.108249$ lbf/in.

Twenty four real and four complex roots were obtained for L_1. The real values for L_1 and the corresponding values of L_2 are shown in Table 2-3. The four complex values for L_1 were $1.2818 \pm 18.2760i$ and $-0.8019 \pm 18.2940i$.

Table 2-3. Twenty-four real solutions.

Case	L_1, inches	L_2, inches
1	-19.0226	-2.5540
2	-16.9651	2.8637
3	-11.7045	0.6215
4	-11.0691	4.3323
5	-11.0649	-4.3325
6	-10.5742	-4.6685

Table 2-3. Continued.

Case	L_1, inches	L_2, inches
7	-10.5716	4.6687
8	-5.1348	10.3382
9	-5.1343	-10.3382
10	-3.7712	-11.2881
11	-3.7704	11.2881
12	-0.02315	11.7686
13	-0.02315	-11.7686
14	3.7733	11.2881
15	3.7737	-11.2881
16	5.1330	-10.3382
17	5.1334	10.3382
18	10.5735	4.6686
19	10.5755	-4.6685
20	11.0631	-4.3326
21	11.0664	4.3324
22	11.7046	0.6193
23	17.4121	2.7902
24	19.5242	-2.4884

At first glance, it appears that a set of input parameters have been found whereby most of the roots of the resulting polynomial equation in L_1 and the corresponding values of L_2 are real. However, from Figure 2-2, it is apparent that the system will be realizable only if

$$|a_{41} - a_{12}| \leq |L_1| \leq a_{41} + a_{12} \, ,$$

$$|a_{34} - a_{23}| \leq |L_1| \leq a_{34} + a_{23} \, ,$$

$$|a_{23} - a_{12}| \leq |L_2| \leq a_{23} + a_{12} \, ,$$

$$|a_{41} - a_{34}| \leq |L_2| \leq a_{41} + a_{34} \, . \tag{2-29}$$

For this numerical case, the system will be realizable only if

$$4.9755 \text{ in.} \leq |L_1| \leq 10.6453 \text{ in.} ,$$

$$4.6684 \text{ in.} \leq |L_2| \leq 10.3382 \text{ in.} \tag{2-30}$$

Table 2-4 lists the cases which satisfy these conditions.

Table 2-4. Eight feasible real solutions.

Case	L_1, inches	L_2, inches
6	-10.5742	-4.6685
7	-10.5716	4.6687
8	-5.1348	10.3382
9	-5.1343	-10.3382
16	5.1330	-10.3382
17	5.1334	10.3382
18	10.5735	4.6686
19	10.5755	-4.6685

The spring L_2 is at an extreme limit value in every one of the cases listed in Table 2-4. When $|L_2| = 10.3382$ in., strut a_{34}, tie a_{41}, and the spring L_2 are collinear. When $|L_2| = 4.668$, strut a_{12}, tie a_{23}, and spring L_2 are collinear. Both configurations are shown in Figure 2-9.
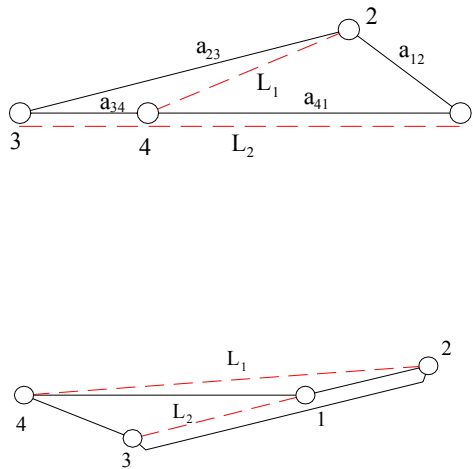
Figure 2-9. Realizable configurations for case 2.

2.3.5 Discussion of Results

It is apparent that obtaining values of L_1 and L_2 that simultaneously satisfy

Equations 2-13 and 2-21 is a necessary condition for equilibrium. Satisfying these

equations is not sufficient, however, to guarantee that the system is physically realizable.

The real values for L_1 and L_2 that were calculated in the second example, but which

violate the conditions of Equation 2-29 are an interesting case. Here some or all the

angles θ_4, θ_4' and θ_4'' will be complex, yet the condition of Equation 2-1 can still be

satisfied.

Presented above was the technique to obtain all possible equilibrium positions of a

planar tensegrity system that incorporates two compliant members. The approach of

satisfying geometric constraints while simultaneously finding positions where the

derivative of the total potential energy with respect to the generalized coordinate equaled

zero resulted in a 28th degree polynomial in a single variable. Although the resulting

polynomial was of higher degree than anticipated, an analysis of the real and complex

solutions indicates that no extraneous solutions were introduced during the variable

elimination procedure. Complex solutions satisfy the geometry equation but they can not

be used to construct the structure, hence they are not the solution.

2.4 Approach 2: Determine (cos θ_4 and cos θ_1) to Minimize Potential Energy

The objective of this approach is to investigate, in closed-form, possible

equilibrium positions using the cosine of two angles, θ_4 and θ_1, as the descriptive

parameters for the system.

Figure 2-10 shows the tensegrity system which is comprised of two struts

(compression members a_{12} and a_{34}), two non-compliant ties (tension members a_{41} and

a_{23}), and two elastic tensile members (springs), one connected between points 1 and 3 and

one between points 2 and 4. It should be noted in Figure 2-10 that strut a_{34} passes

through a slit cut in strut a_{12} and as such the two struts do not intersect or collide.
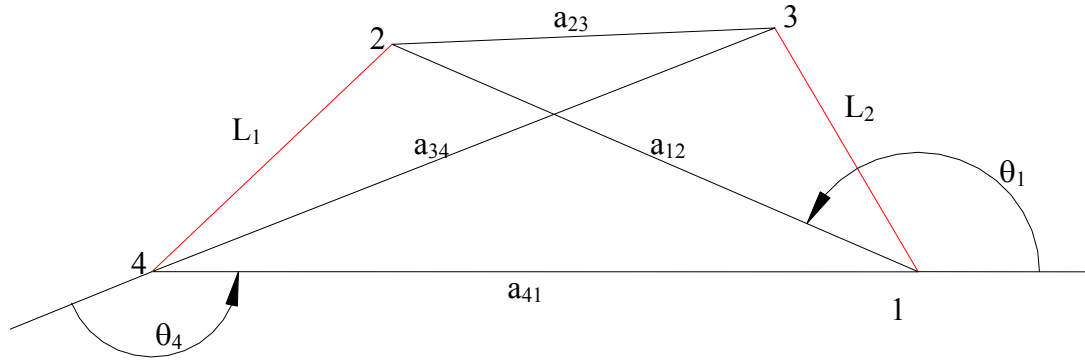
Figure 2-10. Tensegrity system with two struts, two ties and two springs.

The problem statement is written as

given a_{41}, a_{12}, a_{23}, a_{34}

           k_1, k_2, L_{01}, L_{02}

find $\cos\theta_4$ (and corresponding value of $\cos\theta_1$) when the system is in

           equilibrium

The solution approach is outlined as follows:

1. Obtain expression for $\cos\theta_1$ in terms of $\cos\theta_4$

2. Write L_1 in terms of $\cos\theta_1$ and L_2 in terms of $\cos\theta_4$

3. Write the potential energy equation

4. Obtain an equation in the variables $\cos\theta_4$ and $\cos\theta_1$ which corresponds to the

   condition $dU/d\cos\theta_4 = 0$

5. Utilize Sylvester' method to obtain values for $\cos\theta_4$ and $\cos\theta_1$ that satisfy the

   equation from step 1 and step 4.

2.4.1 Obtain Expression for $\cos\theta_1$ in terms of $\cos\theta_4$

The cosine law for the quadrilateral 1-2-3-4 is written as

$$Z_{41} = \frac{a_{23}^2}{2}$$

(2-31)

where by definition

$$Z_{41} = -a_{12}(X_4 s_1 + Y_4 c_1) + Z_4 + \frac{a_{12}^2}{2}$$

(2-32)

$X_4 = a_{34} s_4$ (2-33)

$Y_4 = - (a_{41} + a_{34} c_4)$ (2-34)

$$Z_4 = \frac{a_{34}^2}{2} + \frac{a_{41}^2}{2} + a_{34} a_{41} c_4 \;.$$

(2-35)

Substituting Equation 2-31 into 2-32 and rearranging gives

$$a_{12} X_4 s_1 = -a_{12} Y_4 c_1 + Z_4 + \frac{a_{12}^2}{2} - \frac{a_{23}^2}{2}$$

(2-36)

Substituting for X_4, Y_4, and Z_4 and then squaring this equation and substituting $s_4^2 = 1 - c_4^2$ and $s_1^2 = 1 - c_1^2$ yields

$A c_1^2 + B c_1 + D = 0$ (2-37)

where

$A = A_1 c_4 + A_2$

$A_1 = -2 a_{12}^2 a_{34} a_{41}$

$A_2 = -a_{12}^2 a_{34}^2 - a_{12}^2 a_{41}^2$

$B = B_1 c_4^2 + B_2 c_4 + B_3$

$B_1 = - 2 a_{12} a_{41} a_{34}^2$

$B_2 = a_{12} a_{34} (-a_{34}^2 - a_{12}^2 - 3 a_{41}^2 + a_{23}^2)$

$B_3 = a_{12} a_{41} (-a_{41}^2 - a_{12}^2 - a_{34}^2 + a_{23}^2)$

$$D = D_1\, c_4{}^2 + D_2\, c_4 + D_3$$

$$D_1 = -\, a_{34}{}^2\, a_{41}{}^2 - a_{12}{}^2\, a_{34}{}^2$$

$$D_2 = a_{34}\, a_{41}\, (-a_{41}{}^2 - a_{12}{}^2 - a_{34}{}^2 + a_{23}{}^2)$$

$$D_3 = (-a_{12}{}^2 + a_{23}{}^2 - 2\, a_{12}\, a_{34} - a_{34}{}^2 - a_{41}{}^2)\,(-a_{12}{}^2 + a_{23}{}^2 + 2\, a_{12}\, a_{34} - a_{34}{}^2 - a_{41}{}^2)\,/\,4$$

$$(2.38)$$

Equation (2-37) provides a relationship between c_1 and c_4.

2.4.2: Write L_1 in terms of $\cos\theta_1$ and L_2 in terms of $\cos\theta_4$

A cosine law for triangle 4-1-2 that expresses L_1 in terms of c_1 may be written as

$$\frac{a_{12}{}^2}{2} + \frac{a_{41}{}^2}{2} + a_{12}a_{41}c_1 = \frac{L_1{}^2}{2}.$$

$$(2\text{-}39)$$

A cosine law for the triangle 3-4-1 that expresses L_2 in terms of c_4 may be written as

$$\frac{a_{34}{}^2}{2} + \frac{a_{41}{}^2}{2} + a_{34}a_{41}c_4 = \frac{L_2{}^2}{2}.$$

$$(2\text{-}40)$$

2.4.3: Write the potential energy equation

The energy stored in the springs is given by

$$U = \tfrac{1}{2}\, k_1\, (L_1 - L_{01})^2 + \tfrac{1}{2}\, k_2\, (L_2 - L_{02})^2.$$

$$(2\text{-}41)$$

2.4.4: Express the equation $dU/d\cos\theta_4 = 0$ in terms of the variables c_4 and c_1

The derivative of the potential energy U with respect to c_4 may be written as

$$\frac{dU}{dc_4} = \frac{d\left(\frac{1}{2}k_1(L_1 - L_{01})^2\right)}{dL_1}\frac{dL_1}{dc_1}\frac{dc_1}{dc_4} + \frac{d\left(\frac{1}{2}k_2(L_2 - L_{02})^2\right)}{dL_2}\frac{dL_2}{dc_4}$$

$$(2\text{-}42)$$

$$\frac{dU}{dc_4} = k_1(L_1 - L_{01})\frac{dL_1}{dc_1}\frac{dc_1}{dc_4} + k_2(L_2 - L_{02})\frac{dL_2}{dc_4}$$

$$(2\text{-}43)$$

From Equations 2-39 and 2-40

$$\frac{dL_1}{dc_1} = \frac{a_{12}\,a_{41}}{L_1}$$

(2-44)

$$\frac{dL_2}{dc_4} = \frac{a_{34}\,a_{41}}{L_2}$$

(2-45)

From Equation 2-37

$$\frac{dc_1}{dc_4} = \frac{A_1 c_1^{\,2} + 2B_1 c_1 c_4 + B_2 c_1 + 2D_1 c_4 + D_2}{2A_1 c_1 c_4 + 2A_2 c_1 + B_1 c_4^{\,2} + B_2 c_4 + B_3}$$

(2-46)

Substituting Equations 2-44, 2-45, and 2-46 into 2-43 gives

$$\frac{dU}{dc_4} = k_1(L_1 - L_{01})\left(\frac{a_{12}a_{41}}{L_1}\right)\left(\frac{A_1 c_1^{\,2} + 2B_1 c_1 c_4 + B_2 c_1 + 2D_1 c_4 + D_2}{2A_1 c_1 c_4 + 2A_2 c_1 + B_1 c_4^{\,2} + B_2 c_4 + B_3}\right) + k_2(L_2 - L_{02})\left(\frac{a_{34}a_{41}}{L_2}\right)$$

(2-47)

Equating Equation 2-47 to zero, dividing throughout by a_{41} and rearranging gives

$$k_1(L_1 - L_{01})L_2\,a_{12}\,F + k_2(L_2 - L_{02})L_1\,a_{34}\,G = 0$$

(2-48)

where

$$F = A_1 c_1^{\,2} + 2B_1 c_1 c_4 + B_2 c_1 + 2D_1 c_4 + D_2$$

(2-49)

$$G = 2A_1 c_1 c_4 + 2A_2 c_1 + B_1 c_4^{\,2} + B_2 c_4 + B_3 \qquad .$$

(2-50)

Rearranging this Equation 2-48 gives

$$L_1 L_2 [k_1\,a_{12}\,F + k_2\,a_{34}\,G] = L_1 [k_2\,L_{02}\,a_{34}\,G] + L_2 [k_1\,L_{01}\,a_{12}\,F] \qquad .$$

(2-51)

Squaring both sides and rearranging gives

$$L_1^{\,2} L_2^{\,2} [k_1\,a_{12}\,F + k_2\,a_{34}\,G]^2 - L_1^{\,2}[k_2\,L_{02}\,a_{34}\,G]^2 - L_2^{\,2}[k_1\,L_{01}\,a_{12}\,F]^2 =$$

$$2L_1 L_2 [k_2\,L_{02}\,a_{34}\,G][k_1\,L_{01}\,a_{12}\,F]$$

(2-52)

Squaring both sides again gives

$$L_1^4 L_2^4 [k_1 a_{12} F + k_2 a_{34} G]^4 + L_1^4 [k_2 L_{02} a_{34} G]^4 + L_2^4 [k_1 L_{01} a_{12} F]^4$$
$$-2L_1^4 L_2^2 [k_1 a_{12} F]^2 [k_2 L_{02} a_{34} G]^2 - 2L_1^2 L_2^4 [k_1 a_{12} F]^2 [k_1 L_{01} a_{12} F]^2$$
$$+2L_1^2 L_2^2 [k_2 L_{02} a_{34} G]^2 [k_1 L_{01} a_{12} F]^2$$
$$= 4L_1^2 L_2^2 [k_2 L_{02} a_{34} G]^2 [k_1 L_{01} a_{12} F]^2 \qquad\qquad \text{(2-53)}$$

Rearranging gives,

$$L_1^4 L_2^4 [k_1 a_{12} F + k_2 a_{34} G]^4 + L_1^4 [k_2 L_{02} a_{34} G]^4 + L_2^4 [k_1 L_{01} a_{12} F]^4$$
$$-2L_1^4 L_2^2 [k_1 a_{12} F]^2 [k_2 L_{02} a_{34} G]^2 - 2L_1^2 L_2^4 [k_1 a_{12} F]^2 [k_1 L_{01} a_{12} F]^2$$
$$-2L_1^2 L_2^2 [k_2 L_{02} a_{34} G]^2 [k_1 L_{01} a_{12} F]^2 = 0 \qquad\qquad \text{(2-54)}$$

Substituting for L_1^2 and L_2^2 using Equations 2-39 and 2-40 and factoring the polynomial in terms of c_4 and c_1 gives

$$C_{10} c_1^{10} + C_9 c_1^9 + C_8 c_1^8 + C_7 c_1^7 + C_6 c_1^6 + C_5 c_1^5$$
$$+ C_4 c_1^4 + C_3 c_1^3 + C_2 c_1^2 + C_1 c_1 + C_0 = 0 \qquad\qquad \text{(2-55)}$$

where

$$C_{10} = C_{10,2} c_4^2 + C_{10,1} c_4 + C_{10,0}$$

$$C_9 = C_{9,3} c_4^3 + C_{9,2} c_4^2 + C_{9,1} c_4 + C_{9,0}$$

$$C_8 = C_{8,4} c_4^4 + C_{8,3} c_4^3 + C_{8,2} c_4^2 + C_{8,1} c^4 + C_{8,0}$$

$$C_7 = C_{7,5} c_4^5 + C_{7,4} c_4^4 + C_{7,3} c_4^3 + C_{7,2} c_4^2 + C_{7,1} c^4 + C7_{,0}$$

$$C_6 = C_{6,6} c_4^6 + C_{6,5} c_4^5 + C_{6,4} c_4^4 + C_{6,3} c_4^3 + C_{6,2} c_4^2 + C_{6,1} c4 + C_{6,0}$$

$$C_5 = C_{5,7} c_4^7 + C_{5,6} c_4^6 + C_{5,5} c_4^5 + C_{5,4} c_4^4 + C_{5,3} c_4^3 + C_{5,2} c_4^2 + C_{5,1} c^4 + C_{5,0}$$

$$C_4 = C_{4,8} c_4^8 + C_{4,7} c_4^7 + C_{4,6} c_4^6 + C_{4,5} c_4^5 + C_{4,4} c_4^4 + C_{4,3} c_4^3 + C_{4,2} c_4^2 + C_{4,1} c_4 +$$
$$\qquad C_{4,0}$$

$$C_3 = C_{3,9} c_4^9 + C_{3,8} c_4^8 + C_{3,7} c_4^7 + C_{3,6} c_4^6 + C_{3,5} c_4^5 + C_{3,4} c_4^4 + C_{3,3} c_4^3 + C_{3,2} c_4^2 +$$
$$\qquad C_{3,1} c_4 + C_{3,0}$$

$$C_2 = C_{2,10}\, c_4{}^{10} + C_{2,9}\, c_4{}^9 + C_{2,8}\, c_4{}^8 + C_{2,7}\, c_4{}^7 + C_{2,6}\, c_4{}^6 + C_{2,5}\, c_4{}^5 + C_{2,4}\, c_4{}^4 + C_{2,3}\, c_4{}^3 +$$

$$C_{2,2}\, c_4{}^2 + C_{2,1}\, c_4 + C_{2,0}$$

$$C_1 = C_{1,10}\, c_4{}^{10} + C_{1,9}\, c_4{}^9 + C_{1,8}\, c_4{}^8 + C_{1,7}\, c_4{}^7 + C_{1,6}\, c_4{}^6 + C_{1,5}\, c_4{}^5 + C_{1,4}\, c_4{}^4 + C_{1,3}\, c_4{}^3 +$$

$$C_{1,2}\, c_4{}^2 + C_{1,1}\, c_4 + C_{1,0}$$

$$C_0 = C_{0,10}\, c_4{}^{10} + C_{0,9}\, c_4{}^9 + C_{0,8}\, c_4{}^8 + C_{0,7}\, c_4{}^7 + C_{0,6}\, c_4{}^6 + C_{0,5}\, c_4{}^5 + C_{0,4}\, c_4{}^4 + C_{0,3}\, c_4{}^3 +$$

$$C_{0,2}\, c_4{}^2 + C_{0,1}\, c_4 + C_0 \tag{2-56}$$

All the terms $C_{i,j}$ have been obtained symbolically and are defined in terms of known mechanism parameters. For example,

$$C_{10,0} = 8\, k_1{}^4\, a_{12}{}^6\, a_{41}{}^4\, a_{34}{}^2 A_1{}^4 + 4k_1{}^4\, a_{12}{}^6\, a_{41}{}^6\, A_1{}^4 + 4k_1{}^4\, a_{12}{}^6\, a_{41}{}^2\, a_{34}{}^4\, A_1{}^4$$

$$C_{10,1} = 16k_1{}^4\, a_{12}{}^6\, a_{41}{}^5\, a_{34}\, A_1{}^4 + 16\, k_1{}^4\, a_{12}{}^6\, a_{41}{}^3\, a_{34}{}^3\, A_1{}^4$$

$$C_{10,2} = 16\, k_1{}^4\, a_{12}{}^6\, a_{41}{}^4\, a_{34}{}^2\, A_1{}^4\, c_4{}^2 \tag{2-57}$$

The remaining terms are quite lengthy and are not listed here.

2.4.5 Construct Sylvester Matrix for this approach

Multiplying Equation 2-37 by c_1, $c_1{}^2$, $c_1{}^3$, $c_1{}^4$, $c_1{}^5$, $c_1{}^6$, $c_1{}^7$, $c_1{}^8$, $c_1{}^9$ and 2-55 by c_1 gives 12 equations that can be written as follows

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & B & D \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & B & D & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & A & B & D & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A & B & D & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & A & B & D & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A & B & D & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & A & B & D & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A & B & D & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A & B & D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A & B & D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C_{10} & C_9 & C_8 & C_7 & C_6 & C_5 & C_4 & C_3 & C_2 & C_1 & C_0 \\
C_{10} & C_9 & C_8 & C_7 & C_6 & C_5 & C_4 & C_3 & C_2 & C_1 & C_0 & 0
\end{bmatrix}
\begin{bmatrix}
c_1{}^{11} \\
c_1{}^{10} \\
c_1{}^9 \\
c_1{}^8 \\
c_1{}^7 \\
c_1{}^6 \\
c_1{}^5 \\
c_1{}^4 \\
c_1{}^3 \\
c_1{}^2 \\
c_1 \\
1
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\tag{2-58}
$$

Expansion of the 12×12 determinant yields a 32nd degree polynomial in the parameter c_4.

2.4.6 Numerical Example for Approach 2

Parameters of Example 1 (section 2.3.4) are used for this numerical example:

Strut lengths	$a_{12} = 3$ in.	$a_{34} = 3.5$ in.
Non-compliant tie lengths	$a_{41} = 4$ in.	$a_{23} = 2$ in.
Spring 1 free length & spring constant	$L_{01} = 0.5$ in.	$k_1 = 4$ lbf/in.
Spring 2 free length & spring constant	$L_{02} = 1$ in.	$k_2 = 2.5$ lbf/in.

For this example, a Maple program was written that would solve the determinant of Sylvester matrix in Equation 2-58. The result was a polynomial of degree 32 with respect to the variable $\cos\theta_4$. The solution to the polynomial for this numerical example resulted in 12 real and 20 complex roots. Eight of the real roots were in acceptable range of -1 to +1. For each value of $\cos\theta_4$ the corresponding value of $\cos\theta_1$ was calculated such that Equations 2-37 and 2-55 are simultaneously satisfied. These values, as well as calculated values for L_1 and L_2, are presented in Table 2-5.

Table 2-5.Eight feasible real solutions.

Case	c_4 (radian)	c_1 (radian)	L_1 (inches)	L_2 (inches)
1	-0.8144902900	0.2120649698	-5.4853950884	2.3332963547
2	-0.7083900124	0.1385596475	-5.3221641782	-2.9008756697
3	-0.9290901467	-0.9154296793	-1.7405998094	-1.4951507917
4	-0.8840460933	-0.9381755562	-1.5760033787	1.8699490332
5	-0.9046343262	-0.9312332602	1.6280054527	1.7088706403
6	-0.9434161072	-0.8970754537	1.8628443599	-1.3543814070
7	-0.6228159543	0.0544134519	5.1289299905	-3.2880318246
8	-0.8042767224	0.2077177765	5.4758767915	2.3937944296

Complex values for c_1 that correspond to each complex value of c_4 were determined and are presented in Table 2-6. Although these solutions are not realizable, it is important to show that since the c_4, c_1 pairs satisfy the two equations, no extraneous solutions were introduced in the variable elimination method.

Table 2-6.Twenty complex solutions.

Case	c_4	c_1
1	-1.2513573 - 0.0853382 I	1.0200425 + 0.0625503 I
2	-1.2513573 + 0.0853382 I	1.0200425 - 0.0625503 I
3	-1.2371852 - 0.0586866 I	1.0164651 + 0.0379136 I
4	-1.2371852 + 0.0586866 I	1.0164651 - 0.0379136 I
5	-1.0662774 - 0.0222999 I	-1.0214122 + 0.0110317 I
6	-1.0662774 + 0.0222999 I	-1.0214122 - 0.0110317 I
7	-1.0406024 - 0.0082765 I	-1.0447401 + 0.0104582 I
8	-1.0406024 + 0.0082765 I	-1.0447401 - 0.0104582 I
9	-1.0322412 - 0.0344383 I	-1.0269253 + 0.0377413 I
10	-1.0322412 + 0.0344383 I	-1.0269253 - 0.0377413 I
11	-1.0087191 - 0.0028359 I	-1.1658773 + 0.0338462 I

Table 2-6. Continued.

Case	c_4	c_1
12	-1.0087191 + 0.0028359 I	-1.1658773 - 0.0338462 I
13	0.3360373 - 0.4127430 I	-0.9379955 + 0.6334257 I
14	0.3360373 + 0.4127430 I	-0.9379955 - 0.6334257 I
15	0.3571365 - 0.1068179 I	-0.9796431 + 0.3840155 I
16	0.3571365 + 0.1068179 I	-0.9796431 - 0.3840155 I
17	0.5105685 - 0.0343981 I	-1.0755053 - 0.2990613 I
18	0.5105685 + 0.0343981 I	-1.0755053 + 0.2990613 I
19	0.5418577 - 0.1704979 I	-1.1122907 + 0.4295136 I
20	0.5418577 + 0.1704979 I	-1.1122907 - 0.4295136 I

Lastly, there were four real values of $\cos\theta_4$ that were not in the range $-1 \le \cos\theta_4 \le 1$.

These values are listed in Table 2-7. Corresponding values for $\cos\theta_1$ could not be

obtained that would satisfy both Equations 2-37 and 2-48. Thus it must be concluded that

four extraneous roots were introduced in this second solution approach.

Table 2-7. Four non-feasible real solutions.

Case	c_4 (radian)
1	$-0.7460150\ 10^{18}$
2	-1.0119084
3	-1.0061580
4	$0.7460150\ 10^{18}$

2.4.7 Comparison of Results for Approaches 1 and 2

The numerical example from Section 2.3.4 was examined using this second approach. It was not possible to symbolically expand the determinant of the coefficient matrix in Equation 2-58. Rather the coefficients of the polynomials A, B, D, and C_1 through C_{10} were obtained numerically and then the determinant was expanded to obtain the single polynomial equation in c_4. The numerical example of the second approach resulted in a 32nd degree polynomial in the variable c_4. Table 2-5 shows the resulting values of c_4 that solved this polynomial as well as the corresponding values of c_1 such that Equations 2-37 and 2-55 are simultaneously satisfied. The real solutions are identified in Table 2-5. Values of Table 2-5 were tested in the differential of potential energy to identify equilibrium cases. Analysis showed that cases 3, 4, 5, and 6 of Table 2-5 represent equilibrium configurations. These cases are identically similar to cases presented in Table 2-1.

Comparison of the results of the numerical example using the two approaches outlined in this chapter showed that both approaches agree on the correct numerical answer. Hence, both approaches are valid in finding equilibrium configuration were closed form solution for this structure can not be found.

CHAPTER 3
THREE SPRING PLANAR TENSEGRITY

3.1 Introduction

This chapter considers the case of a planar tensegrity system with one non-compliant member, a_{41}, three compliant members, springs L_1, L_2, L_3, and two struts, a_{34} and a_{12} (see Figure 3-1). In Figure 3-1 the two struts do not intersect but one passes through a slit cut in the other one. The objective of this study is to find all possible equilibrium configurations in stable condition (minimum potential energy) when given the lengths of the struts and the non-compliant tie together with the free length and spring constants for the three compliant members.

The device shown in Figure 3-1 is a two degree of freedom system. Two parameters must be specified, in addition to the constant mechanism parameters, in order to define the configuration of the device. These two parameters will be referred to as the *descriptive parameters* for the system. One obvious set of descriptive parameters are the angles θ_4 and θ_1. Considering the non-compliant member a_{41} as being fixed to ground, specification of θ_4 will define the location of point 3. Similarly, specification of θ_1 will define the location of point 2.

Two approaches to solve this problem are presented in this chapter. Both aim to find a set of descriptive parameters that minimize the potential energy in the system. In the first approach, the lengths of the compliant members L_1 and L_2 are chosen as the descriptive parameters. Derivatives of the potential energy equation are obtained with respect to L_1 and L_2 and values for the descriptive parameters are obtained such that these

derivatives are zero, corresponding to either a minimum or maximum potential energy state. In the second approach, the cosines of the angles θ_4 and θ_1 were chosen as the descriptive parameters. The cosines of the angles were chosen rather than the angles themselves in the hope that the resulting equations would be simpler in that, for example, a single value of $\cos\theta_4$ accounts for the obvious symmetry in solutions that will occur with respect to the fixed member a_{41}.
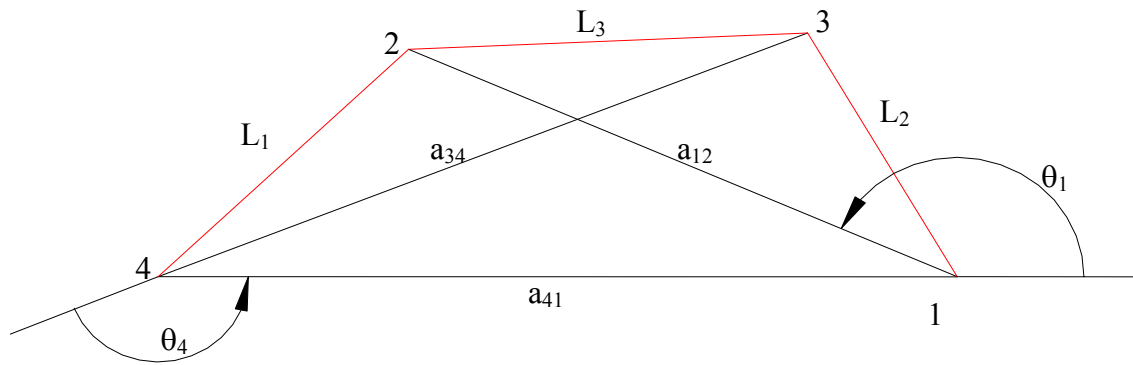
Figure 3-1. Tensegrity structure with two struts, a non-compliant (tie), and three springs.

3.2 First Approach Problem Statement (Descriptive Parameters L₁, L₂)

The problem statement can be explicitly written as:

Given: a_{12}, a_{34} lengths of struts,

a_{41} length of non-compliant tie

k_1, L_{01} spring constant and free length of compliant tie between points 4 and 2

k_2, L_{02} spring constant and free length of compliant tie between points 3 and 1

k_3, L_{03} spring constant and free length of compliant tie between points 2 and 3

Find: L_1 length of spring 1 at equilibrium position,

L_2 length of spring 2 at equilibrium position,

L_3 corresponding length of spring 3 at equilibrium position

3.2.1 Development of Geometric Constraint Equation

Figure 3-2 shows the nomenclature that is used. L_1, L_2, and L_3 are the extended

lengths of the compliant ties between points 4 and 2, points 3 and 1, and points 2 and 3.
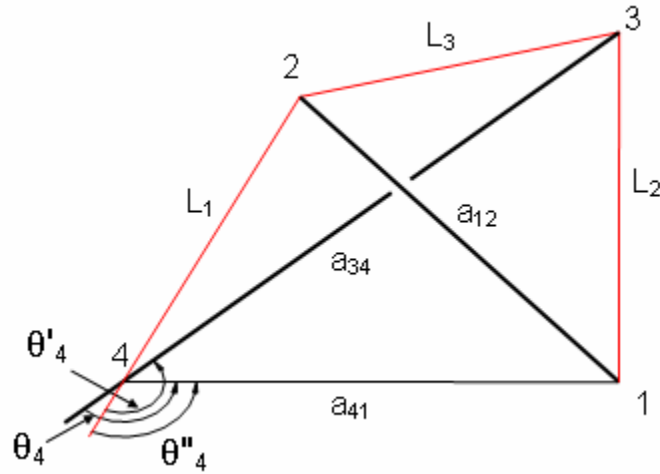
Figure 3-2. Planar tensegrity structure.

The analysis starts by considering different triangles in this structure. Figure 3-3 shows

the triangle formed by side a_{34}, L_3, and L_1.
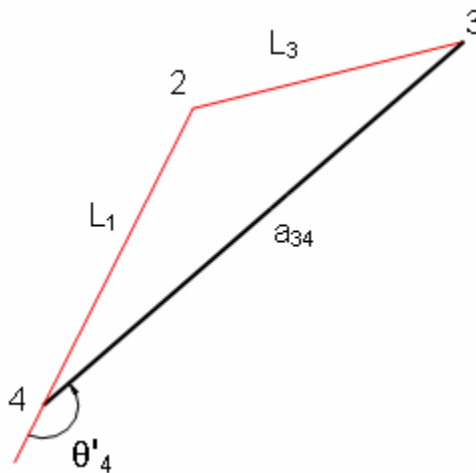
Figure 3-3. Triangle 4-3-2.

A cosine law for this triangle can be written as

$$\frac{L_1^2}{2} + \frac{a_{34}^2}{2} + L_1 a_{34} \cos\theta_4' = \frac{L_3^2}{2}.$$

(3-1)

Solving for $\cos\theta_4'$ yields

$$\cos\theta_4' = \frac{L_3^2 - L_1^2 - a_{34}^2}{2 L_1 a_{34}}.$$

(3-2)

Figure 3-4 shows the triangle formed by side a_{41}, a_{12}, and L_1. A cosine law for this triangle can be written as

$$\frac{L_1^2}{2} + \frac{a_{41}^2}{2} + L_1 a_{41} \cos\theta_4'' = \frac{a_{12}^2}{2}$$

(3-3)

Solving for $\cos\theta_4''$ yields

$$\cos\theta_4'' = \frac{a_{12}^2 - L_1^2 - a_{41}^2}{2 L_1 a_{41}}.$$

(3-4)
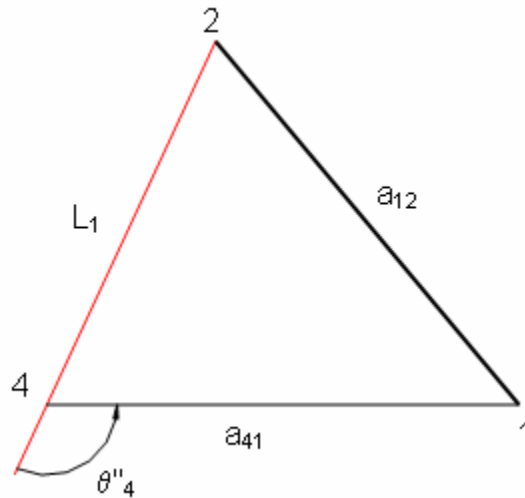
Figure 3-4. Triangle 4-1-2.

Figure 3-5 shows the triangle formed by a_{41}, a_{34}, and L_2. A cosine law for this triangle can be written as

$$\frac{a_{34}^2}{2} + \frac{a_{41}^2}{2} + a_{34} a_{41} \cos\theta_4 = \frac{L_2^2}{2}.$$

(3-5)
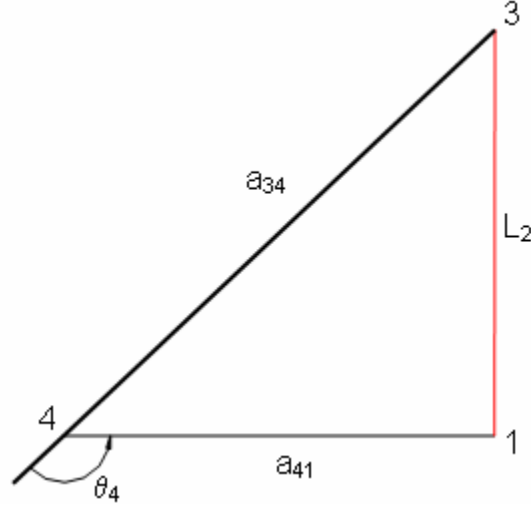
Figure 3-5. Triangle 4-1-3.

Solving for $\cos\theta_4$ yields

$$\cos\theta_4 = \frac{L_2^2 - a_{34}^2 - a_{41}^2}{2 a_{34} a_{41}}.$$

(3-6)

From Figure 3-2 it is apparent that

$$\theta_4 + \theta_4' = \pi + \theta_4''$$

(3-7)

Equating the cosine of the left and right sides of Equation 3-7 yields

$$\cos(\theta_4 + \theta_4') = \cos(\pi + \theta_4'')$$

(3-8)

and expanding this Equation 3-8 yields

$$\cos\theta_4 \cos\theta_4' - \sin\theta_4 \sin\theta_4' = -\cos\theta_4''.$$

(3-9)

Rearranging (3-9) gives

$$\cos\theta_4 \cos\theta_4' + \cos\theta_4'' = \sin\theta_4 \sin\theta_4'.$$

(3-10)

Squaring both sides yields

$$(\cos\theta_4)^2 \, (\cos\theta_4')^2 + 2 \, \cos\theta_4 \, \cos\theta_4' \, \cos\theta_4'' + (\cos\theta_4'')^2 = (\sin\theta_4)^2 \, (\sin\theta_4')^2 \quad (3\text{-}11)$$

Substituting for $(\sin\theta_4)^2$ and $(\sin\theta_4')^2$ in terms of $\cos\theta_4$ and $\cos\theta_4'$ gives

$$(\cos\theta_4)^2 \, (\cos\theta_4')^2 + 2 \, \cos\theta_4 \, \cos\theta_4' \, \cos\theta_4'' + (\cos\theta_4'')^2 = (1\text{-}\cos^2\theta_4) \, (1\text{-}\cos^2\theta_4') \quad (3\text{-}12)$$

Equations 3-2, 3-4, and 3-6 are substituted into Equation 3-12 to yield a single equation in the parameters L_1, L_2, and L_3 which can be written as

$$G_1 \, L_3{}^4 + (G_2 \, L_2{}^2 + G_3) \, L_3{}^2 + (G_4 \, L_2{}^4 + G_5 \, L_2{}^2 + G_6) = 0 \quad (3\text{-}13)$$

where

$$G_1 = a_{41}{}^2,$$

$$G_2 = G_{2a} \, L_1{}^2 + G_{2b},$$

$$G_3 = G_{3a} \, L_1{}^2 + G_{3b},$$

$$G_4 = G_{4a} \, L_1{}^2,$$

$$G_5 = G_{5a} \, L_1{}^4 + G_{5b} \, L_1{}^2 + G_{5c},$$

$$G_6 = G_{6a} \, L_1{}^2 + G_{6b} \, . \quad (3\text{-}14)$$

and where

$$G_{2a} = -1, \qquad\qquad\qquad G_{2b} = a_{12}{}^2 - a_{41}{}^2,$$

$$G_{3a} = (a_{34}{}^2 - a_{41}{}^2), \qquad G_{3b} = a_{41}{}^2 \, (a_{41}{}^2 - a_{12}{}^2 - a_{34}{}^2) - a_{12}{}^2 \, a_{34}{}^2,$$

$$G_{4a} = 1,$$

$$G_{5a} = 1, \qquad\qquad G_{5b} = (\text{-}a_{12}{}^2 - a_{34}{}^2 - a_{41}{}^2), \qquad G_{5c} = a_{34}{}^2 \, (a_{41}{}^2 - a_{12}{}^2),$$

$$G_{6a} = a_{12}{}^2 \, (a_{41}{}^2 - a_{34}{}^2), \qquad G_{6b} = a_{12}{}^2 \, a_{34}{}^2 \, (a_{12}{}^2 + a_{34}{}^2 - a_{41}{}^2) \, . \quad (3\text{-}15)$$

3.2.2 Development of Potential Energy Equations

The potential energy of the system can be written as

$$U = \frac{1}{2}k_1 \, (L_1 - L_{01})^2 + \frac{1}{2}k_2 \, (L_2 - L_{02})^2 + \frac{1}{2}k_3 \, (L_3 - L_{03})^2 \quad (3\text{-}16)$$

At equilibrium, the potential energy will be a minimum. This condition can be determined as the configuration of the mechanism whereby the derivative of the potential energy taken with respect to the descriptive parameters L_1 and L_2 both equal zero. The geometric constraint equation, Equation 3-13, contains three unknown terms, L_1, L_2, and L_3. From this equation, L_3 can be considered as a dependent variable of L_1 and L_2. The following two expressions may be written:

$$\frac{\partial U}{\partial L_1} = k_1 (L_1 - L_{01}) + k_3 (L_3 - L_{03}) \frac{\partial L_3}{\partial L_1} = 0 \ , \tag{3-17}$$

$$\frac{\partial U}{\partial L_2} = k_2 (L_2 - L_{02}) + k_3 (L_3 - L_{03}) \frac{\partial L_3}{\partial L_2} = 0 \ . \tag{3-18}$$

The derivatives $\partial L_3/\partial L_1$ and $\partial L_3/\partial L_2$ can be determined via implicit differentiation from Equation 3-13 as

$$\frac{\partial L_3}{\partial L_1} = \frac{-L_1 [2 L_1^2 L_2^2 + G_{2a} L_2^2 L_3^2 + G_{3a} L_3^2 + L_2^4 + G_{5b} L_2^2 + G_{6a}]}{L_3 [2 G_1 L_3^2 + G_{2a} L_1^2 L_2^2 + G_{2b} L_2^2 + G_{3a} L_1^2 + G_{3b}]} , \tag{3-19}$$

$$\frac{\partial L_3}{\partial L_2} = \frac{-L_2 [2 L_1^2 L_2^2 + G_{2a} L_1^2 L_3^2 + G_{2b} L_3^2 + L_1^4 + G_{5b} L_1^2 + G_{5c}]}{L_3 [2 G_1 L_3^2 + G_{2a} L_1^2 L_2^2 + G_{2b} L_2^2 + G_{3a} L_1^2 + G_{3b}]} . \tag{3-20}$$

Substituting (3-19) into (3-17) and rearranging gives

$$(D_1 L_2^2 + D_2) L_3^3 + (D_3 L_2^2 + D_4) L_3^2 + (D_5 L_2^4 + D_6 L_2^2 + D_7) L_3 + (D_8 L_2^4 + D_9 L_2^2 + D_{10}) = 0 \tag{3-21}$$

where

$D_1 = D_{1a} L_1$,

$D_2 = D_{2a} L_1 + D_{2b}$,

$D_3 = D_{3a} L_1$,

$D_4 = D_{4a} L_1$,

$D_5 = D_{5a} L_1$,

$$D_6 = D_{6a} \, L_1{}^3 + D_{6b} \, L_1{}^2 + D_{6c} \, L_1 + D_{6d} \ ,$$

$$D_7 = D_{7a} \, L_1{}^3 + D_{7b} \, L_1{}^2 + D_{7c} \, L_1 + D_{7d} \ ,$$

$$D_8 = D_{8a} \, L_1 \ ,$$

$$D_9 = D_{9a} \, L_1{}^3 + D_{9b} \, L_1$$

$$D_{10} = D_{10a} \, L_1 \quad (3.22)$$

and

$$D_{1a} = - \, G_{2a} \, k_3 \ ,$$

$$D_{2a} = 2 \, G_1 \, k_1 - G_{3a} \, k_3, \quad D_{2b} = - \, 2 \, G_1 \, k_1 \, L_{01} \ ,$$

$$D_{3a} = G_{2a} \, k_3 \, L_{03} \ ,$$

$$D_{4a} = G_{3a} \, k_3 \, L_{03} \ ,$$

$$D_{5a} = - \, k_3 \ ,$$

$$D_{6a} = G_{2a} \, k_1 - 2 \, k_3 \ , \quad D_{6b} = -G_{2a} \, k_1 \, L_{01} \ , \quad D_{6c} = G_{2b} \, k_1 - G_{5b} \, k_3 \ , \quad D_{6d} = -G_{2b} \, k_1 \, L_{01} \ ,$$

$$D_{7a} = G_{3a} \, k_1 \ , \quad D_{7b} = -G_{3a} \, k_1 \, L_{01} \ , \quad D_{7c} = G_{3b} \, k_1 - G_{6a} \, k_3 \ , \quad D_{7d} = -G_{3b} \, k_1 \, L_{01} \ ,$$

$$D_{8a} = k_3 \, L_{03} \ ,$$

$$D_{9a} = 2 \, k_3 \, L_{03} \ , \qquad D_{9b} = G_{5b} \, k_3 \, L_{03}$$

$$D_{10a} = G_{6a} \, k_3 \, L_{03} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3\text{-}23)$$

Substituting (3-20) into (3-18) and rearranging gives

$$(E_1 \, L_2 + E_2) \, L_3{}^3 + (E_3 \, L_2) \, L_3{}^2 + (E_4 \, L_2{}^3 + E_5 \, L_2{}^2 {}_+ E_6 \, L_2 + E_7) \, L_3 + (E_8 \, L_2{}^3 + E_9 \, L_2) = 0$$

$$(3\text{-}24)$$

where

$$E_1 = E_{1a} \, L_1{}^2 + E_{1b} \ ,$$

$$E_2 = - \, 2 \, G_1 \, k_2 \, L_{02} \ ,$$

$$E_3 = E_{3a} \, L_1{}^2 + E_{3b} \ ,$$

$$E_4 = E_{4a} \, L_1{}^2 + E_{4b} \ ,$$

$$E_5 = E_{5a} L_1^2 + E_{5b} ,$$

$$E_6 = E_{6a} L_1^4 + E_{6b} L_1^2 + E_{6c} ,$$

$$E_7 = E_{7a} L_1^2 + E_{7b} ,$$

$$E_8 = E_{8a} L_1^2 ,$$

$$E_9 = E_{9a} L_1^4 + E_{9b} L_1^2 + E_{9c} \qquad (3\text{-}25)$$

and

$$E_{1a} = -G_{2a} k_3 , \qquad E_{1b} = -G_{2b} k_3 + 2 G_1 k_2 ,$$

$$E_{3a} = G_{2a} k_3 L_{03} , \qquad E_{3b} = G_{2b} k_3 L_{03} ,$$

$$E_{4a} = G_{2a} k_2 - 2 k_3 , \qquad E_{4b} = G_{2b} k_2 ,$$

$$E_{5a} = -G_{2a} k_2 L_{02} , \qquad E_{5b} = -G_{2b} k_2 L_{02} ,$$

$$E_{6a} = - k_3 , \qquad E_{6b} = G_{3a} k_2 - G_{5b} k_3 , \quad E_{6c} = G_{3b} k_2 - G_{5c} k_3 ,$$

$$E_{7a} = -G_{3a} k_2 L_{02} , \qquad E_{7b} = -G_{3b} k_2 L_{02} ,$$

$$E_{8a} = 2 k_3 L_{03} ,$$

$$E_{9a} = k_3 L_{03} , \qquad E_{9b} = G_{5b} k_3 L_{03} , \qquad E_{9c} = G_{5c} k_3 L_{03} . \qquad (3\text{-}26)$$

3.2.3 Create Solution Matrix

Equations 3-13, 3-21, and 3-24 are three equations in the three unknowns L_1, L_2, and L_3. Sylvester's method, reference 11, is applied in order to obtain sets of values for these parameters that simultaneously satisfy all three equations. In this solution, the parameter L_1 is embedded in the coefficients of the three equations to yield three equations in the apparent unknowns L_2 and L_3. Determining the condition that these new coefficients (which contain L_1) must satisfy such that the three equations can have common roots for L_2 and L_3 will yield a single polynomial in L_1.

Equation 3-13 was multiplied by L_2, L_3, L_2L_3, L_3^2, L_2^2, $L_2L_3^2$, $L_2^2L_3$, $L_2^2L_3^2$, L_3^3, L_2^3, $L_2L_3^3$, $L_2^2L_3^3$, $L_2^3L_3$, $L_2^3L_3^2$, and $L_2^3L_3^3$. Equation 3-21 was multiplied by L_3, L_2, L_3^2, $L_2L_3^2$, L_3^3, L_2^2, $L_2L_3^3$, $L_2^2L_3$, $L_2^2L_3^2$, L_2^3, L_3^4, $L_2^3L_3$, $L_2^3L_3^2$, and $L_2L_3^4$. Equation 3-24 was multiplied by L_2, L_3, L_2L_3, L_3^2, L_2^2, $L_2L_3^2$, $L_2^2L_3$, $L_2^2L_3^2$, L_3^3, L_2^3, $L_2L_3^3$, $L_2^2L_3^3$, $L_2^3L_3$, $L_2^3L_3^2$, L_2^4, L_3^4, $L_2^4L_3$, $L_2^4L_3^2$, $L_2L_3^4$, and $L_2^2L_3^4$. This resulted in a set of 52 equations that can be written in matrix for as

$$\mathbf{M}\,\lambda = \mathbf{0} \, . \tag{3-27}$$

The vector λ is written as

$$\lambda = [L_2^7L_3^3, \ L_2^5L_3^5, \ L_2^3L_3^7, \ L_2^7L_3^2, \ L_2^6L_3^3, \ L_2^5L_3^4, \ L_2^4L_3^5, \ L_2^3L_3^6, \ L_2^2L_3^7, \ L_2^7L_3,$$
$$L_2^6L_3^2, \ L_2^5L_3^3, \ L_2^4L_3^4, \ L_2^3L_3^5, \ L_2^2L_3^6, \ L_2L_3^7, \ L_2^7, \ L_2^6L_3, \ L_2^5L_3^2, \ L_2^4L_3^3, \ L_2^3L_3^4,$$
$$L_2^2L_3^5, \ L_2L_3^6, \ L_3^7, \ L_2^6, \ L_2^5L_3, \ L_2^4L_3^2, \ L_2^3L_3^3, \ L_2^2L_3^4, \ L_2L_3^5, \ L_3^6, \ L_2^5, \ L_2^4L_3,$$
$$L_2^3L_3^2, \ L_2^2L_3^3, \ L_2L_3^4, \ L_3^5, \ L_2^4, \ L_2^3L_3, \ L_2^2L_3^2, \ L_2L_3^3, \ L_3^4, \ L_2^3, \ L_2^2L_3, \ L_2L_3^2, \ L_3^3,$$
$$L_2^2, \ L_2L_3, \ L_3^2, \ L_2, \ L_3, \ 1]^{\mathrm{T}} \, . \tag{3-28}$$

The coefficient matrix $\mathbf{M}$ is a $52{\times}52$ matrix. To show this $52{\times}52$ coefficient matrix, it is subdivided into the four $26{\times}26$ submatrices $\mathbf{M}_{11}$, $\mathbf{M}_{12}$, $\mathbf{M}_{21}$, $\mathbf{M}_{22}$ as

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \, . \tag{3-29}$$

The four sub matrices are:

$$
\mathbf{M}_{11} =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_5 & 0 & D_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_5 & 0 & D_1 & 0 & 0 & 0 & 0 & D_8 & 0 & D_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_4 & 0 & E_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_4 & 0 & E_1 & 0 & 0 & 0 & 0 & 0 & E_8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_4 & 0 & E_1 & 0 & 0 & 0 & 0 & E_8 & E_5 & E_3 & E_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_4 & 0 & G_2 & 0 & G_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_4 & 0 & G_2 & 0 & G_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_4 & 0 & G_2 & 0 & G_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_5 & 0 & D_1 & 0 & 0 & 0 & 0 & D_8 & 0 & D_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_5 & 0 & D_1 & 0 & 0 & 0 & 0 & 0 & D_8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & D_5 & 0 & D_1 & 0 & 0 & 0 & D_8 & 0 & D_3 & 0 & 0 & 0 & 0 & 0 & 0 & D_6 & 0 & D_2 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\mathbf{M}_{21} =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_5 & 0 & D_1 & 0 & 0 & 0 & 0 & D_8 & 0 & D_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & D_5 & 0 & D_1 & 0 & 0 & 0 & D_8 & 0 & D_3 & 0 & 0 & 0 & 0 & 0 & 0 & D_6 & 0 & D_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_4 & 0 & E_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_4 & 0 & E_1 & 0 & 0 & 0 & 0 & 0 & E_8 & E_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_4 & 0 & E_1 & 0 & 0 & 0 & 0 & E_8 & E_5 & E_3 & E_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & E_4 & 0 & E_1 & 0 & 0 & 0 & E_8 & E_5 & E_3 & E_2 & 0 & 0 & 0 & 0 & 0 & E_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_4 & 0 & E_1 & 0 & 0 & 0 & 0 & E_8 & E_5 & E_3 & E_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & E_4 & 0 & E_1 & 0 & 0 & 0 & E_8 & E_5 & E_3 & E_2 & 0 & 0 & 0 & 0 & E_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_4 & 0 & G_2 & 0 & G_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_4 & 0 & G_2 & 0 & G_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_4 & 0 & G_2 & 0 & G_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & G_4 & 0 & G_2 & 0 & G_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_5 & 0 & G_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_4 & 0 & G_2 & 0 & G_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_5 \\
0 & 0 & 0 & G_4 & 0 & G_2 & 0 & G_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_5 & 0 & G_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
G_4 & G_2 & G_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_5 & 0 & G_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_5 & 0 & D_1 & 0 & 0 & 0 & 0 & D_8 & 0 & D_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_6 \\
0 & 0 & 0 & 0 & 0 & 0 & D_5 & 0 & D_1 & 0 & 0 & 0 & D_8 & 0 & D_3 & 0 & 0 & 0 & 0 & 0 & 0 & D_6 & 0 & D_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & D_5 & 0 & D_1 & 0 & 0 & 0 & D_8 & 0 & D_3 & 0 & 0 & 0 & 0 & 0 & 0 & D_6 & 0 & D_2 & 0 & 0 & 0 & 0 & 0 & 0 & D_9 \\
D_5 & D_1 & 0 & D_8 & 0 & D_3 & 0 & 0 & 0 & 0 & 0 & D_6 & 0 & D_2 & 0 & 0 & 0 & 0 & D_9 & 0 & D_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & D_5 & D_1 & 0 & 0 & D_8 & 0 & D_3 & 0 & 0 & 0 & 0 & 0 & D_6 & 0 & D_2 & 0 & 0 & 0 & 0 & D_9 & 0 & D_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_4 & 0 & E_1 & 0 & 0 & 0 & 0 & E_8 & E_5 & E_3 & E_2 & 0 & 0 & 0 & 0 & 0 & 0 & E_6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_4 & 0 & E_1 & 0 & 0 & 0 & 0 & E_8 & E_5 & E_3 & E_2 & 0 & 0 & 0 \\
0 & 0 & 0 & E_4 & 0 & E_1 & 0 & 0 & 0 & E_8 & E_5 & E_3 & E_2 & 0 & 0 & 0 & 0 & 0 & E_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_9 \\
E_4 & E_1 & 0 & E_8 & E_5 & E_3 & E_3 & 0 & 0 & 0 & 0 & E_6 & 0 & 0 & 0 & 0 & 0 & 0 & E_9 & E_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & E_4 & 0 & E_1 & 0 & 0 & 0 & E_8 & E_5 & E_3 & E_2 & 0 & 0 & 0 & 0 & 0 & E_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & E_4 & E_1 & 0 & 0 & E_8 & E_5 & E_3 & E_2 & 0 & 0 & 0 & 0 & E_6 & 0 & 0 & 0 & 0 & 0 & 0 & E_9 & E_7 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\mathbf{M}_{12} =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_4 & 0 & G_2 & 0 & G_1 & 0 & 0 & 0 & 0 & G_5 & 0 & G_3 & 0 & 0 & G_6 \\
0 & 0 & 0 & 0 & 0 & 0 & D_5 & 0 & D_1 & 0 & 0 & D_8 & 0 & D_3 & 0 & 0 & 0 & D_6 & 0 & D_2 & D_9 & 0 & D_4 & 0 & D_7 & D_{10} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_4 & 0 & E_1 & 0 & E_8 & E_5 & E_3 & E_2 & 0 & E_6 & 0 & E_9 & E_7 & 0 \\
D_5 & 0 & D_1 & 0 & 0 & 0 & D_8 & 0 & D_3 & 0 & 0 & 0 & 0 & D_6 & 0 & D_2 & 0 & D_9 & 0 & D_4 & 0 & 0 & D_7 & 0 & D_{10} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & E_4 & 0 & E_1 & 0 & 0 & E_8 & E_5 & E_3 & E_2 & 0 & 0 & E_6 & 0 & 0 & E_9 & E_7 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & E_4 & 0 & E_1 & 0 & 0 & E_8 & E_5 & E_3 & E_2 & 0 & 0 & E_6 & 0 & 0 & E_9 & E_7 & 0 & 0 & 0 \\
E_4 & 0 & E_1 & 0 & 0 & 0 & E_8 & E_5 & E_3 & E_2 & 0 & 0 & 0 & E_6 & 0 & 0 & 0 & E_9 & E_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & G_4 & 0 & G_2 & 0 & G_1 & 0 & 0 & 0 & 0 & 0 & 0 & G_5 & 0 & G_3 & 0 & 0 & 0 & 0 & G_6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & G_4 & 0 & G_2 & 0 & G_1 & 0 & 0 & 0 & 0 & 0 & 0 & G_5 & 0 & G_3 & 0 & 0 & 0 & 0 & G_6 & 0 \\
0 & G_2 & 0 & G_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_5 & 0 & G_3 & 0 & 0 & 0 & 0 & 0 & 0 & G_6 & 0 & 0 & 0 & 0 & 0 \\
0 & D_1 & 0 & 0 & 0 & D_8 & 0 & D_3 & 0 & 0 & 0 & 0 & D_6 & 0 & D_2 & 0 & D_9 & 0 & D_4 & 0 & 0 & D_7 & 0 & D_{10} & 0 & 0 \\
D_8 & 0 & D_3 & 0 & 0 & 0 & 0 & 0 & D_6 & 0 & D_2 & 0 & 0 & D_9 & 0 & D_4 & 0 & 0 & 0 & D_7 & 0 & 0 & D_{10} & 0 & 0 & 0 \\
0 & D_6 & 0 & D_2 & 0 & 0 & 0 & D_9 & 0 & D_4 & 0 & 0 & 0 & 0 & D_7 & 0 & 0 & 0 & D_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & E_4 & 0 & E_1 & 0 & 0 & 0 & E_8 & E_5 & E_3 & E_2 & 0 & 0 & 0 & E_6 & 0 & 0 & 0 & E_9 & E_7 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & E_1 & 0 & 0 & 0 & E_8 & E_5 & E_3 & E_2 & 0 & 0 & 0 & E_6 & 0 & 0 & 0 & E_9 & E_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
E_8 & E_5 & E_3 & E_2 & 0 & 0 & 0 & 0 & E_6 & 0 & 0 & 0 & 0 & E_9 & E_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
E_5 & E_3 & E_2 & 0 & 0 & 0 & 0 & E_6 & 0 & 0 & 0 & 0 & E_9 & E_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & E_6 & 0 & 0 & 0 & 0 & E_9 & E_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
G_4 & 0 & G_2 & 0 & G_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_5 & 0 & G_3 & 0 & 0 & 0 & 0 & 0 & 0 & G_6 & 0 & 0 & 0 \\
G_2 & 0 & G_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_5 & 0 & G_3 & 0 & 0 & 0 & 0 & 0 & 0 & G_6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & G_5 & 0 & G_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & G_5 & 0 & G_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
G_5 & 0 & G_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & D_6 & 0 & D_2 & 0 & 0 & 0 & D_9 & 0 & D_4 & 0 & 0 & 0 & 0 & D_7 & 0 & 0 & 0 & D_{10} & 0 & 0 & 0 & 0 & 0 & 0 \\
D_3 & 0 & 0 & 0 & 0 & 0 & D_6 & 0 & D_2 & 0 & 0 & D_9 & 0 & D_4 & 0 & 0 & 0 & D_7 & 0 & 0 & D_{10} & 0 & 0 & 0 & 0 & 0 \\
0 & D_9 & 0 & D_4 & 0 & 0 & 0 & 0 & 0 & D_7 & 0 & 0 & 0 & 0 & D_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\mathbf{M}_{22} =
\begin{bmatrix}
D_6 & 0 & D_2 & 0 & 0 & 0 & D_9 & 0 & D_4 & 0 & 0 & 0 & 0 & D_7 & 0 & 0 & 0 & D_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
D_9 & 0 & D_4 & 0 & 0 & 0 & 0 & 0 & D_7 & 0 & 0 & 0 & 0 & D_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & E_8 & E_5 & E_3 & E_2 & 0 & 0 & 0 & 0 & E_6 & 0 & 0 & 0 & 0 & E_9 & E_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
E_3 & E_2 & 0 & 0 & 0 & 0 & E_6 & 0 & 0 & 0 & 0 & E_9 & E_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & E_6 & 0 & 0 & 0 & 0 & 0 & E_9 & E_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & E_9 & E_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
E_6 & 0 & 0 & 0 & 0 & 0 & E_9 & E_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
E_9 & E_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_5 & 0 & G_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & G_5 & 0 & G_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & G_5 & 0 & G_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & G_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & G_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & D_2 & 0 & 0 & 0 & D_9 & 0 & D_4 & 0 & 0 & 0 & 0 & D_7 & 0 & 0 & 0 & D_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & D_9 & 0 & D_4 & 0 & 0 & 0 & 0 & 0 & D_7 & 0 & 0 & 0 & 0 & D_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & D_4 & 0 & 0 & 0 & 0 & 0 & D_7 & 0 & 0 & 0 & 0 & D_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & D_7 & 0 & 0 & 0 & 0 & 0 & D_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & D_7 & 0 & 0 & 0 & 0 & 0 & D_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & E_9 & E_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & E_6 & 0 & 0 & 0 & 0 & 0 & E_9 & E_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
E_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & E_9 & E_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

(3-30)

In order for a solution for the vector λ to exist, it is necessary for the 52 resulting equations to be linearly dependent. This will occur if the determinant of **M** equals zero.

It was not possible to symbolically expand the determinant of matrix **M**. A numerical case was analyzed and a polynomial of degree 158 in the variable L_1 was obtained. It was not possible to solve this high degree polynomial for the values of L_1, although several commercial and in-house written algorithms were attempted. Because of this, a different method was attempted to solve the set of equations 3-13, 3-21, and 3-24.

3.2.4 Solution of Three Simultaneous Equations in Three Unknowns – Continuation Method

The continuation method (Morgan [18], Mora [19], Cullen [20], Strang [21], Garcia and Li [22], Morgan [23-24], Wampler et al., [25]) which is a numerical technique to solve a set of equations in multiple variables are used. This is as opposed to Sylvester's method which would lead to a symbolic solution of the problem.

A concise description of the continuity method is presented by Tsai, 1999. Suppose one wishes to solve the set of equations F(**x**) which are defined by

$$F(\mathbf{x}): \begin{cases} f_1(x_1, x_2, \cdots, x_n) = 0 \\ f_2(x_1, x_2, \cdots, x_n) = 0 \\ \quad\quad\vdots \\ f_n(x_1, x_2, \cdots, x_n) = 0 \end{cases} \tag{3-31}$$

F(**x**) is called the target system.

The continuation method begins by first estimating the total number of possible solution sets (sets of values for L_1, L_2, and L_3 for our case) that satisfy the given equations. For example, Bezout's theorem states that a polynomial of total degree n has at most n isolated solutions in the complex Euclidean space. Including solutions at

infinity, the Bezout number of a polynomial system is equal to the total degree of the system.

Next, an initial system, $G(\mathbf{x}) = 0$, is obtained, whose solution will be of the same degree as that of $F(\mathbf{x})$, but whose solution set is known in closed form. In other words, $G(\mathbf{x})$ maintains the same polynomial structure as $F(\mathbf{x})$.

Finally, a homotopy function $H(\mathbf{x}, t)$ is prepared such as

$$H(\mathbf{x}, t) = \gamma\,(1\text{-}t)\,G(\mathbf{x}) + t\,F(\mathbf{x}) \tag{3-32}$$

where γ is a random complex constant. When $t=0$, the homotopy function equals the initial system, $G(\mathbf{x})$. When $t=1$, the homotopy function equals the target system, $F(\mathbf{x})$. Recall that the solutions to $G(\mathbf{x})$ are known. As the parameter t is increased in small steps from 0 to 1, the solutions of $H(\mathbf{x}, t)$ can be tracked (referred to as path tracking) and when $t = 1$, these solutions will be the solutions to the original target system. If the degree of the solution set was overestimated, some of the solutions will track to infinity and these can easily be discarded.

3.2.5 Numerical Example

The following information is given:

strut lengths:

$a_{12} = 14$ in. $a_{34} = 12$ in.

non-compliant tie lengths:

$a_{41} = 10$ in.

spring 1 free length & spring constant:

$L_{01} = 8$ in. $k_1 = 1$ lbf/in.

spring 2 free length & spring constant:

$L_{02} = 2$ in. $k_2 = 2.687$ lbf/in.

spring 3 free length & spring constant:

$$L_{03} = 2.5 \text{ in.} \quad k_3 = 3.465 \text{ lbf/in.}$$

Based on these values, the coefficients in equations 3-13, 3-21, and 3-24 were evaluated to yield the three equations

$$100 \, L_3{}^4 + [(-L_1{}^2 + 96) \, L_2{}^2 + 44 \, L_1{}^2 - 52224] \, L_3{}^2 + L_1{}^2 \, L_2{}^4$$

$$+ (L_1{}^4 - 440 \, L_1{}^2 - 13824) \, L_2{}^2 - 8624 \, L_1{}^2 + 6773760 = 0 \tag{3-33}$$

$$(2.5 \, L_1 \, L_2{}^2 + 90 \, L_1 - 1600) \, L_3{}^3 + (-8.663 \, L_1 \, L_2{}^2 + 381.165 \, L_1) \, L_3{}^2 + [-2.5 \, L_1 \, L_2{}^4$$

$$+ (-6 \, L_1{}^3 + 8 \, L_1{}^2 + 1196 \, L_1 - 768) \, L_2{}^2 + 44 \, L_1{}^3 - 352 \, L_1{}^2 - 30664 \, L_1 + 417792] \, L_3$$

$$+ 8.663 \, L_1 \, L_2{}^4 + (17.326 \, L_1{}^3 - 3811.651 \, L_1) \, L_2{}^2 - 74708.354 \, L_1 = 0 \tag{3-34}$$

$$[(2.5 \, L_1{}^2 + 160) \, L_2 - 1074.637] \, L_3{}^3 + (831.633 - 8.663 \, L_1{}^2) \, L_2 \, L_3{}^2$$

$$+ [(-7 \, L_1{}^2 + 192) \, L_2{}^3 + (-515.826 + 5.373 \, L_1{}^2) \, L_2{}^2 + (-2.5 \, L_1{}^4 + 1188 \, L_1{}^2 - 69888) \, L_2$$

$$- 236.420 \, L_1{}^2 + 280609.161] \, L_3 + 17.326 \, L_1{}^2 \, L_2{}^3$$

$$+ (-119755.135 + 8.663 \, L_1{}^4 - 3811.651 \, L_1{}^2) \, L_2 = 0 \,. \tag{3-35}$$

The continuation method was run on this set of three equations in three unknowns to obtain all solution sets for the three spring lengths, L_1, L_2, and L_3, for the particular numerical example. The software PHCpack (Verschelde [26]) was used to implement the method.

The PHCpack software estimated the number of possible solutions to be 136. Seven real solutions were obtained and these are listed in Table 3-1.

Table 3-1: Seven real solutions for three-spring planar tensegrity system(units are inches)

Case	L_1	L_2	L_3
1	13.000	8.000	7.017
2	-11.376	-10.371	-5.333
3	-7.585	9.097	10.106
4	-11.029	12.557	-3.044
5	13.969	-5.800	9.164
6	14.248	-9.373	-4.774
7	13.181	11.599	-2.488

An equilibrium analysis was conducted for the seven cases and only the first case was indeed in equilibrium. Case 1 is shown in Figure 3-6.
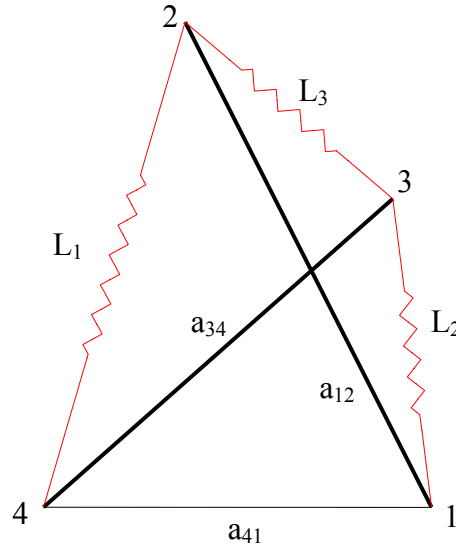
Figure 3-6. Case 1, equilibrium solution.

3.3 Second Approach Problem Statement (calculate cosθ₄, cosθ₁)

In the second approach, the cosines of the angles θ_4 and θ_1 were chosen as the descriptive parameters. The cosines of the angles were chosen rather than the angles themselves in the hope that the resulting equations would be simpler in that, for example, a single value of $\cos\theta_4$ accounts for the obvious symmetry in solutions that will occur with respect to the fixed member a_{41}.

The problem statement is presented as follows:

Given: Strut lengths a_{12}, a_{34}, tie length a_{41}, and the spring constants and free

lengths k_1, k_2, k_3, L_{01}, L_{02}, L_{03}.

Find: $\cos\theta_4$ and $\cos\theta_1$ when the system is in equilibrium.

The solution approach is as follows:

1. Obtain expressions for L_1, L_2, and L_3 in terms of $\cos\theta_4$ and $\cos\theta_1$

2. Write the potential energy equation

3. Determine values of $\cos\theta_4$ and $\cos\theta_1$ such that $dU/d\cos\theta_4 = dU/d\cos\theta_1 = 0$

3.3.1 Development of Geometric Equations

Figure 3-1 shows the nomenclature used. Springs L_1, L_2, and L_3 are the extended lengths of the compliant ties between points 4 and 2, 2 and 3, and 1 and 3 respectively. The unloaded lengths of springs are given by L_{01}, L_{02}, and L_{03}. A cosine law for the quadrilateral 1-2-3-4 can be written as:

$$Z_{41} = \frac{L_3^{\,2}}{2} \qquad (3\text{-}36)$$

where,

$$Z_{41} = -a_{12}(X_4 s_1 + Y_4 c_1) + Z_4 + \frac{a_{12}^{\,2}}{2} \qquad (3\text{-}37)$$

and where

$$X_4 = a_{34}\, s_4 \qquad (3\text{-}38)$$

$$Y_4 = -\,(a_{41} + a_{34}\, c_4) \qquad (3\text{-}39)$$

$$Z_4 = \frac{a_{34}^{\,2}}{2} + \frac{a_{41}^{\,2}}{2} + a_{34} a_{41} c_4 \; . \qquad (3\text{-}40)$$

Substituting (3-37) into (3-36) and rearranging gives

$$a_{12}X_4 s_1 = -a_{12}Y_4 c_1 + Z_4 + \frac{a_{12}^2}{2} - \frac{L_3^2}{2} \ . \tag{3-41}$$

Substituting (3-38), (3-39), and (3-40) into (3-41), then squaring it, and substituting for $s_4^2 = 1 - c_4^2$ and $s_1^2 = 1 - c_1^2$ and multiplying the entire equation by 4 yields

$$L_3^4 + A\ L_3^2 + B = 0 \tag{3-42}$$

where

$$A = A_1\ c_4 + A_2\ c_1 + A_3\ c_1\ c_4 + A_4\ , \tag{3-43}$$

$$B = B_1\ c_1^2 + B_2\ c_1^2\ c_4 + B_3\ c_1 + B_4\ c_1\ c_4 + B_5\ c_1\ c_4^2 + B_6\ c_4^2 + B_7\ c_4 + B_8 \tag{3-44}$$

and where

$A_1 = -\ 4\ a_{34}\ a_{41}$

$A_2 = -\ 4\ a_{12}\ a_{41}$

$A_3 = -\ 4\ a_{12}\ a_{34}$

$A_4 = -\ 2\ (a_{12}^2 + a_{34}^2 + a_{41}^2)$

$B_1 = 4\ a_{12}^2\ (a_{34}^2 + a_{41}^2)$

$B_2 = 8\ a_{12}^2\ a_{34}\ a_{41}$

$B_3 = 4\ a_{12}\ a_{41}\ (a_{41}^2 + a_{12}^2 + a_{34}^2)$

$B_4 = 4\ a_{12}\ a_{34}\ (a_{34}^2 + a_{12}^2 + 3\ a_{41}^2)$

$B_5 = 8\ a_{12}\ a_{41}\ a_{34}^2$

$B_6 = 4\ a_{34}^2\ (a_{12}^2 + a_{41}^2)$

$B_7 = 4\ a_{34}\ a_{41}\ (a_{41}^2 + a_{34}^2 + a_{12}^2)$

$B_8 = (a_{41}^2 + a_{34}^2 - 2\ a_{12}\ a_{34} + a_{12}^2)\ (a_{41}^2 + a_{34}^2 + 2\ a_{12}\ a_{34} + a_{12}^2)\ . \tag{3-45}$

Equation 3-42 expresses L_3 as a function of c_4 and c_1. A cosine law for triangle 4-1-2 may be written as

$$\frac{a_{12}^{2}}{2} + \frac{a_{41}^{2}}{2} + a_{12}a_{41}c_{1} = \frac{L_{1}^{2}}{2} \ . \qquad (3\text{-}46)$$

A cosine law for the triangle 3-4-1 may be written as

$$\frac{a_{34}^{2}}{2} + \frac{a_{41}^{2}}{2} + a_{34}a_{41}c_{4} = \frac{L_{2}^{2}}{2} \ . \qquad (3\text{-}47)$$

Equations 3-42, 3-46, and 3-47 define the three spring lengths in terms of the variables $\cos\theta_4$ and $\cos\theta_1$.

3.3.2 Development of Potential Energy Equations

The total potential energy stored in all three springs is given by,

$$U = \tfrac{1}{2} k_1 (L_1\text{-}L_{01})^2 + \tfrac{1}{2} k_2 (L_2\text{-}L_{02})^2 + \tfrac{1}{2} k_3 (L_3\text{-}L_{03})^2 \ . \qquad (3\text{-}48)$$

Differentiating the potential energy with respect to c_4 and c_1 and then evaluating values for c_4 and c_1 that cause the derivative of the potential energy to equal zero, will identify configurations of either minimum or maximum potential energy. These derivatives may be written as,

$$dU/dc_4 = k_1 (L_1 - L_{01}) \, dL_1/dc_4 + k_2 (L_2 - L_{02}) \, dL_2/dc_4 + k_3 (L_3 - L_{03}) \, dL_3/dc_4$$

$$(3\text{-}49)$$

$$dU/dc_1 = k_1 (L_1 - L_{01}) \, dL_1/dc_1 + k_2 (L_2 - L_{02}) \, dL_2/dc_1 + k_3 (L_3 - L_{03}) \, dL_3/dc_1$$

$$(3\text{-}50)$$

Since from 3-46, L_1 is not a function of c_4, $dL_1/dc_4 = 0$. Similarly, from 3-47, L_2 is not a function of c_1 and thus $dL_2/dc_1 = 0$. Equations 3-49 and 3-50 reduce to

$$dU/dc_4 = k_2 (L_2 - L_{02}) \, dL_2/dc_4 + k_3 (L_3 - L_{03}) \, dL_3/dc_4 \ , \qquad (3\text{-}51)$$

$$dU/dc_1 = k_1 (L_1 - L_{01}) \, dL_1/dc_1 + k_3 (L_3 - L_{03}) \, dL_3/dc_1 \ . \qquad (3\text{-}52)$$

The term dL_2/dc_4 is evaluated from 3-47 as

$$\frac{dL_2}{dc_4} = \frac{a_{34}a_{41}}{L_2} \tag{3-53}$$

and the term dL_1/dc_1 is evaluated from 3-46 as

$$\frac{dL_1}{dc_1} = \frac{a_{12}a_{41}}{L_1}. \tag{3-54}$$

Implicit differentiation of 3-42 for L_3 with respect to c_4 and c_1 yields

$$\frac{dL_3}{dc_4} = -\frac{(A_1 + A_3 c_1)L_3{}^2 + B_2 c_1{}^2 + B_4 c_1 + 2B_5 c_1 c_4 + 2B_6 c_4 + B_7}{2L_3(2L_3{}^2 + A_1 c_4 + A_2 c_1 + A_3 c_1 c_4 + A_4)} \tag{3-55}$$

$$\frac{dL_3}{dc_1} = -\frac{(A_2 + A_3 c_4)L_3{}^2 + 2B_1 c_1 + 2B_2 c_1 c_4 + B_3 + B_4 c_4 + B_5 c_4{}^2}{2L_3(2L_3{}^2 + A_1 c_4 + A_2 c_1 + A_3 c_1 c_4 + A_4)}. \tag{3-56}$$

Substituting 3-53 and 3-55 into 3-51 and 3-54 and 3-56 into 3-52 and equating to zero

yields

$$(M_1 L_3{}^3 + M_2 L_3{}^2 + M_3 L_3 + M_4) L_2 + M_5 L_3{}^3 + M_6 L_3 = 0 \tag{3-57}$$

$$(N_1 L_3{}^3 + N_2 L_3{}^2 + N_3 L_3 + N_4) L_1 + N_5 L_3{}^3 + N_6 L_3 = 0 \tag{3-58}$$

Where the coefficients M_i and N_i, $i=1..6$, are functions of c_4 and c_1 as

$M_1 = -k_3 (A_3 c_1 + A_1) + 4 k_2 a_{34} a_{41}$

$M_2 = k_3 L_{03} (A_3 c_1 + A_1)$

$M_3 = -k_3 B_2 c_1{}^2 + [2 (k_2 a_{34} a_{41}A_3 - k_3 B_5) c_4 + 2 k_2 a_{34} a_{41} A_2 - k_3 B_4) c_1$

$\qquad + 2 (k_2 a_{34} a_{41} A_1 - k_3 B_6) c_4 - k_3 B_7 + 2 k_2 a_{34} a_{41} A_4$

$M_4 = k_3 L_{03} B_2 c_1{}^2 + k_3 L_{03} B_4 c_1 + 2 k_3 L_{03} B_5 c_1 c_4 + 2 k_3 L_{03} B_6 c_4 + k_3 L_{03} B_7$

$M_5 = -4 k_2 a_{34} a_{41} L_{02}$

$M_6 = -2 k_2 (a_{34} a_{41}L_{02} A_2 c_1 + a_{34} a_{41} L_{02} A_3 c_1 c_4 + a_{34} a_{41} L_{02} A_1 c_4$

$\qquad + a_{34} a_{41} L_{02} A_4) \tag{3-59}$

$N_1 = - k_3 (A_3 c_4 + A_2) + 4 k_1 a_{12} a_{41}$

$N_2 = k_3 L_{03} (A_3 c_4 + A_2)$

$$N_3 = 2 \, k_1 \, a_{12} \, a_{41} \, (A_1 \, c_4 + A_2 \, c_1 + A_3 \, c_1 \, c_4 + A_4) - k_3 \, (B_4 \, c_4 + B_5 \, c_4{}^2 + 2 \, B_1 \, c_1$$

$$+ \, 2 \, B_2 \, c_1 \, c_4 + B_3)$$

$$N_4 = k_3 \, L_{03} \, (B_4 \, c_4 + B_5 \, c_4{}^2 + 2 \, B_1 \, c_1 + B_2 \, c_1 \, c_4 + B_3)$$

$$N_5 = - \, 4 \, k_1 \, a_{12} \, a_{41} \, L_{01}$$

$$N_6 = -2 \, k_1 \, a_{12} \, a_{41} \, L_{01} \, (A_4 + A_1 \, c_4 + A_2 \, c_1 + A_3 \, c_1 \, c_4) \,. \tag{3-60}$$

Equations 3-57 and 3-58 may be written as

$$(M_1 \, L_3{}^3 + M_2 \, L_3{}^2 + M_3 \, L_3 + M_4) \, L_2 = - \, M_5 \, L_3{}^3 - M_6 \, L_3 \,, \tag{3-61}$$

$$(N_1 \, L_3{}^3 + N_2 \, L_3{}^2 + N_3 \, L_3 + N_4) \, L_1 = - \, N_5 \, L_3{}^3 - N_6 \, L_3 \,. \tag{3-62}$$

Squaring both sides of both equations gives

$$(M_1 \, L_3{}^3 + M_2 \, L_3{}^2 + M_3 \, L_3 + M_4)^2 \, L_2{}^2 = (M_5 \, L_3{}^3 + M_6 \, L_3)^2 \,, \tag{3-63}$$

$$(N_1 \, L_3{}^3 + N_2 \, L_3{}^2 + N_3 \, L_3 + N_4)^2 \, L_1{}^2 = (N_5 \, L_3{}^3 + N_6 \, L_3)^2 \,. \tag{3-64}$$

Using 3-46 and 3-47 to substitute for $L_1{}^2$ and $L_2{}^2$ will yield two equations in the

parameters c_1, c_4, and L_3. These two equations can be arranged as

$$(p_1 \, c_1{}^2 + p_2 \, c_1 + p_3) \, c_4{}^3 + (p_4 \, c_1{}^3 + p_5 \, c_1{}^2 + p_6 \, c_1 + p_7) \, c_4{}^2$$

$$+ \, (p_8 \, c_1{}^4 + p_9 \, c_1{}^3 + p_{10} \, c_1{}^2 + p_{11} \, c_1 + p_{12}) \, c_4$$

$$+ \, (p_{13} \, c_1{}^4 + p_{14} \, c_1{}^3 + p_{15} \, c_1{}^2 + p_{16} \, c_1 + p_{17}) = 0 \tag{3-65}$$

$$(q_1 \, c_1 + q_2) \, c_4{}^4 + (q_3 \, c_1{}^2 + q_4 \, c_1 + q_5) \, c_4{}^3$$

$$+ \, (q_6 \, c_1{}^3 + q_7 \, c_1{}^2 + q_8 \, c_1 + q_9) \, c_4{}^2 + (q_{10} \, c_1{}^3 + q_{11} \, c_1{}^2 + q_{12} \, c_1 + q_{13}) \, c_4$$

$$+ \, (q_{14} \, c_1{}^3 + q_{15} \, c_1{}^2 + q_{16} \, c_1 + q_{17}) = 0 \tag{3-66}$$

where

$$p_1 = p_{1a} \, L_3{}^2 + p_{1b} \, L_3 + p_{1c}$$

$$p_2 = p_{2a} \, L_3{}^2 + p_{2b} \, L_3 + p_{2c}$$

$$p_3 = p_{3a} \, L_3{}^2 + p_{3b} \, L_3 + p_{3c}$$

$$p_4 = p_{4a} \, L_3{}^2 + p_{4b} \, L_3 + p_{4c}$$

$$p_5 = p_{5a} \, L_3{}^4 + p_{5b} \, L_3{}^3 + p_{5c} \, L_3{}^2 + p_{5d} \, L_3 + p_{5e}$$

$$p_6 = p_{6a} \, L_3{}^4 + p_{6b} \, L_3{}^3 + p_{6c} \, L_3{}^2 + p_{6d} \, L_3 + p_{6e}$$

$$p_7 = p_{7a} \, L_3{}^4 + p_{7b} \, L_3{}^3 + p_{7c} \, L_3{}^2 + p_{7d} \, L_3 + p_{7e}$$

$$p_8 = p_{8a} \, L_3{}^2 + p_{8b} \, L_3 + p_{8c}$$

$$p_9 = p_{9a} \, L_3{}^4 + p_{9b} \, L_3{}^3 + p_{9c} \, L_3{}^2 + p_{9d} \, L_3 + p_{9e}$$

$$p_{10} = p_{10a} \, L_3{}^6 + p_{10b} \, L_3{}^5 + p_{10c} \, L_3{}^4 + p_{10d} \, L_3{}^3 + p_{10e} \, L_3{}^2 + p_{10f} \, L_3 + p_{10g}$$

$$p_{11} = p_{11a} \, L_3{}^6 + p_{11b} \, L_3{}^5 + p_{11c} \, L_3{}^4 + p_{11d} \, L_3{}^3 + p_{11e} \, L_3{}^2 + p_{11f} \, L_3 + p_{11g}$$

$$p_{12} = p_{12a} \, L_3{}^6 + p_{12b} \, L_3{}^5 + p_{12c} \, L_3{}^4 + p_{12d} \, L_3{}^3 + p_{12e} \, L_3{}^2 + p_{12f} \, L_3 + p_{12g}$$

$$p_{13} = p_{13a} \, L_3{}^2 + p_{13b} \, L_3 + p_{13c}$$

$$p_{14} = p_{14a} \, L_3{}^4 + p_{14b} \, L_3{}^3 + p_{14c} \, L_3{}^2 + p_{14d} \, L_3 + p_{14e}$$

$$p_{15} = p_{15a} \, L_3{}^6 + p_{15b} \, L_3{}^5 + p_{15c} \, L_3{}^4 + p_{15d} \, L_3{}^3 + p_{15e} \, L_3{}^2 + p_{15f} \, L_3 + p_{15g}$$

$$p_{16} = p_{16a} \, L_3{}^6 + p_{16b} \, L_3{}^5 + p_{16c} \, L_3{}^4 + p_{16d} \, L_3{}^3 + p_{16e} \, L_3{}^2 + p_{16f} \, L_3 + p_{16g}$$

$$p_{17} = p_{17a} \, L_3{}^6 + p_{17b} \, L_3{}^5 + p_{17c} \, L_3{}^4 + p_{17d} \, L_3{}^3 + p_{17e} \, L_3{}^2 + p_{17f} \, L_3 + p_{17g} \qquad (3\text{-}67)$$

$$q_1 = q_{1a} \, L_3{}^2 + q_{1b} \, L_3 + q_{1c}$$

$$q_2 = q_{2a} \, L_3{}^2 + q_{2b} \, L_3 + q_{2c}$$

$$q_3 = q_{3a} \, L_3{}^2 + q_{3b} \, L_3 + q_{3c}$$

$$q_4 = q_{4a} \, L_3{}^4 + q_{4b} \, L_3{}^3 + q_{4c} \, L_3{}^2 + q_{4d} \, L_3 + q_{4e}$$

$$q_5 = q_{5a} \, L_3{}^4 + q_{5b} \, L_3{}^3 + q_{5c} \, L_3{}^2 + q_{5d} \, L_3 + q_{5e}$$

$$q_6 = q_{6a} \, L_3{}^2 + q_{6b} \, L_3 + q_{6c}$$

$$q_7 = q_{7a} \, L_3{}^4 + q_{7b} \, L_3{}^3 + q_{7c} \, L_3{}^2 + q_{7d} \, L_3 + q_{7e}$$

$$q_8 = q_{8a} \, L_3{}^6 + q_{8b} \, L_3{}^5 + q_{8c} \, L_3{}^4 + q_{8d} \, L_3{}^3 + q_{8e} \, L_3{}^2 + q_{8f} \, L_3 + q_{8g}$$

$$q_9 = q_{9a} \, L_3{}^6 + q_{9b} \, L_3{}^5 + q_{9c} \, L_3{}^4 + q_{9d} \, L_3{}^3 + q_{9e} \, L_3{}^2 + q_{9f} \, L_3 + q_{9g}$$

$$q_{10} = q_{10a} L_3{}^2 + q_{10b} L_3 + q_{10c}$$

$$q_{11} = q_{11a} L_3{}^4 + q_{11b} L_3{}^3 + q_{11c} L_3{}^2 + q_{11d} L_3 + q_{11e}$$

$$q_{12} = q_{12a} L_3{}^6 + q_{12b} L_3{}^5 + q_{12c} L_3{}^4 + q_{12d} L_3{}^3 + q_{12e} L_3{}^2 + q_{12f} L_3 + q_{12g}$$

$$q_{13} = q_{13a} L_3{}^6 + q_{13b} L_3{}^5 + q_{13c} L_3{}^4 + q_{13d} L_3{}^3 + q_{13e} L_3{}^2 + q_{13f} L_3 + q_{13g}$$

$$q_{14} = q_{14a} L_3{}^2 + q_{14b} L_3 + q_{14c}$$

$$q_{15} = q_{15a} L_3{}^4 + q_{15b} L_3{}^3 + q_{15c} L_3{}^2 + q_{15d} L_3 + q_{15e}$$

$$q_{16} = q_{16a} L_3{}^6 + q_{16b} L_3{}^5 + q_{16c} L_3{}^4 + q_{16d} L_3{}^3 + q_{16e} L_3{}^2 + q_{16f} L_3 + q_{16g}$$

$$q_{17} = q_{17a} L_3{}^6 + q_{17b} L_3{}^5 + q_{17c} L_3{}^4 + q_{17d} L_3{}^3 + q_{17e} L_3{}^2 + q_{17f} L_3 + q_{17g} \ . \qquad (3\text{-}68)$$

The coefficients p_{1a} through q_{17g} are functions of the given constant parameters. These coefficients are defined in the Appendix.

The coefficients p_{1a} through q_{17g} have been obtained symbolically. For example, the terms p_{1a} through p_{1c} are written as

$$p_{1a} = 8 \, k_2{}^2 \, a_{34}{}^3 \, a_{41}{}^3 \, A_3{}^2 - 16 \, k_2 \, a_{34}{}^2 \, a_{41}{}^2 \, A_3 \, k_3 \, B_5 + 8 \, k_3{}^2 \, B_5{}^2 \, a_{34} \, a_{41}$$

$$p_{1b} = \text{-}16 \, k_3{}^2 \, B_5{}^2 \, L_{03} \, a_{34} \, a_{41} + 16 \, k_2 \, a_{34}{}^2 \, a_{41}{}^2 \, A_3 \, k_3 \, L_{03} \, B_5$$

$$p_{1c} = 8 \, k_3{}^2 \, L_{03}{}^2 \, B_5{}^2 \, a_{34} \, a_{41}$$

The remaining terms are not listed here due to their complexity.

Similarly, equation 3-42 is an equation in the same parameters, c_1, c_4, and L_3. This equation can be factored as

$$(r_1 \, c_1 + r_2) \, c_4{}^2 + (r_3 \, c_1{}^2 + r_4 \, c_1 + r_5) \, c_4 + (r_6 \, c_1{}^2 + r_7 \, c_1 + r_8) = 0 \qquad (3\text{-}69)$$

where

$$r_1 = 8 \, a_{12} \, a_{41} \, a_{34}{}^2$$

$$r_2 = 4 \, a_{34}{}^2 \, (a_{12}{}^2 + a_{41}{}^2)$$

$$r_3 = 8 \, a_{34} \, a_{41} \, a_{12}{}^2$$

$$r_4 = (-4\ a_{12}\ a_{34})\ L_3^2 + 4\ a_{12}\ a_{34}\ (a_{34}^2 + a_{12}^2 + 3\ a_{41}^2)$$

$$r_5 = (-4\ a_{34}\ a_{41})\ L_3^2 + 4\ a_{34}\ a_{41}\ (a_{34}^2 + a_{12}^2 + a_{41}^2)$$

$$r_6 = 4\ a_{12}^2\ (a_{34}^2 + a_{41}^2)$$

$$r_7 = (-4\ a_{12}\ a_{41})\ L_3^2 + 4\ a_{12}\ a_{41}\ (a_{41}^2 + a_{34}^2 + a_{12}^2)$$

$$r_8 = L_3^4 - 2\ (a_{34}^2 + a_{12}^2 + a_{41}^2)\ L_3^2 \tag{3-70}$$

3.3.3 Solution of Equation Set for c_4, c_1, and L_3

Equations 3-65, 3-66, and 3-69 are factored such that the parameter L_3 is embedded in the coefficients p_1 through r_8. Sylvester's method was used in an attempt to solve these equations for all possible sets of values of L_3, c_1, and c_4. Equation 3-65 was multiplied by c_1, c_1^2, c_4, c_4^2, c_4^3, c_1^4, $c_{12}c_4$, $c_1c_4^2$, and $c_1^2c_4^2$ to obtain 10 equations (including itself). Equation 3-66 was multiplied by c_1, c_1^2, c_1^3, c_4, c_4^2, c_1c_4, $c_1^2c_4$, $c_1^3c_4$, $c_1c_4^2$ and $c_1^2c_4^2$ to obtain 11 equations (including itself). Equation 3-69 was multiplied by c_1, c_1^2, c_1^3, c_1^4, c_4, c_4^2, c_4^3, c_4^4, c_1c_4, $c_1^2c_4$, $c_1^3c_4$, $c_1^4c_4$, $c_1c_4^2$, $c_1^2c_4^2$, $c_1^3c_4^2$, $c_1^4c_4^2$, $c_1c_4^3$, $c_1^2c_4^3$, $c_1^3c_1^4$, $c_1c_4^4$ and $c_1^2c_4^4$ to obtain 22 equations including itself. This resulted in a set of 43 "homogeneous" equations in 43 unknowns.

The condition for a solution to this set of equations was that they are linearly dependent, i.e. the determinant of the 43×43 coefficient matrix must equal zero. This would yield a polynomial in the single variable L_3. Due to the complexity of the problem, it was not possible to expand this determinant symbolically. Similarly, it was not possible to root the polynomial that resulted from a particular numerical example. Because of this, the Continuation Method was used again to determine solutions for the same numerical example presented in Section 3.2.5.

Substituting the given numerical values into equations 3-65, 3-66, and 3-69 are

yield three equations for this numerical example. For this case, the coefficients p_{1a}

through q_{17g} and r_1 through r_8 are evaluated and presented in Appendix b.

The continuation method was run on this set of three equations in three unknowns

to obtain all solution sets for the three variables, L_3, c_1, and c_4, for the particular

numerical example. The software PHCpack (Verschelde [26]) was again used to

implement the method.

The PHCpack software estimated the number of possible solutions to be 136.

Seven real solutions were obtained and these are listed in Table 3-2.

Table 3-2. Real value answers using cosine of θ_1, θ_4 in radians and L_3 in inches.

Case	Cosine θ_1	Cosine θ_4	L_3
1	-0.453571523	-0.749999925	7.01658755
2	-0.594965126	-0.568509675	-5.33336481
3	-0.851695156	-0.671871253	10.1059910
4	-0.622722694	-0.359683532	-3.04401046
5	-0.360201938	-0.876513972	9.16391281
6	-0.332122619	-0.650641695	-4.77378239
7	-.0436671807	-0.456122910	-2.48758125

The lengths of the springs L_1 and L_2 were calculated for each of the seven cases and

it was determined that the seven real solutions here correspond directly with the seven

real solutions that were determined from the first problem formulation and listed in Table

3-1. As before, case 1 was the only case that was indeed in equilibrium and the

mechanism is shown in this configuration in Figure 3-6.

3.4 Conclusion

Two approaches were presented to solve the three-spring planar tensegrity

equilibrium problem. Both approaches resulted in high degree polynomials. These

polynomials could not be derived symbolically and were in fact difficult to derive numerically.

A continuation approach was presented to solve the set of three equations for each of the two solution approaches. A force balance analysis was conducted to identify all realizable solutions; in this case one.

It is apparent that further analysis of Sylvester's variable elimination procedure may be necessary to reduce the number of simultaneous equations, and thereby the size of the univariate polynomial that results. As a result of the work in this chapter it can be said that the degree of the solution has been bounded, although further work may yield a simpler result.

CHAPTER 4
FOUR SPRING PLANAR TENSEGRITY

4.1 Introduction

In this chapter planar tensegrity mechanisms with two struts and four elastic ties

(see Figure 4-1) are analyzed to determine all possible equilibrium configurations for the

device when no external forces or moments are applied. The stable equilibrium position

is determined by identifying the configurations at which the potential energy stored in the

four springs is a minimum.
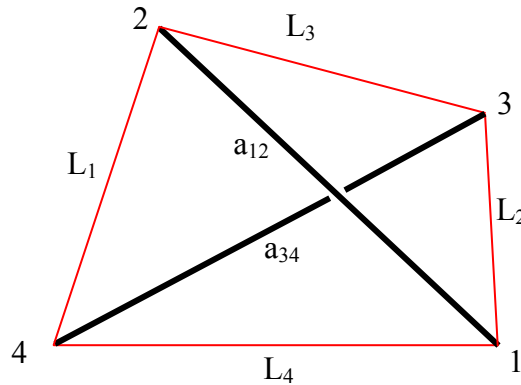
Figure 4-1. Two strut, four spring planar tensegrity device

Four non-linear equations are obtained in the variables L_1, L_2, L_3, and L_4 which

correspond to the equilibrium lengths of the four compliant ties. One equation is

associated with the geometry constraint that must be satisfied in order for the mechanism

to be assembled. The other three equations correspond to the derivative of the potential

energy equation taken with respect to three of the compliant tie lengths.

Solution of these four equations in order to find sets of values for the parameters

L_1 through L_4 that simultaneously satisfy all four equations proved to be a very difficult

process. For this case, the continuity method was used in an attempt to find all

equilibrium solutions for two numerical cases. The results of the continuity method are

then checked to first verify that the set of solutions simultaneously satisfy all four

equations and then to determine whether each solution set represents a minimum

potential energy and an equilibrium state.

4.2 Problem Statement

The problem statement can be explicitly written as:

Given

a_{12}, a_{34}	lengths of struts,
k_1, L_{01}	spring constant and its free length between points 4 and 2,
k_2, L_{02}	spring constant and its free length between points 3 and 1,
k_3, L_{03}	spring constant and its free length between points 3 and 2,
k_4, L_{04}	spring constant and its free length between points 4 and 1.

Find at equilibrium position

L_1	length of spring 1,
L_2	length of spring 2,
L_3	length of spring 3,
L_4	length of spring 4,

It should be noted that the problem statement could be formulated in a variety of

ways. The solution presented here uses L_1, L_2, and L_3 as the three generalized

parameters, knowledge of which completely describes the system (i.e. the value for L_4,

the last remaining parameter, can be deduced). The geometry equation that relates L_4 in

terms of the three generalized parameters is derived first. Following this, a derivative of

the potential energy equation is taken with respect to each of the three generalized

parameters to yield an additional three equations.

4.3 Development of Geometric and Potential Energy Constraint Equations

The geometry equation is derived in a manner similar to that presented in Section 3.2.1. The parameter a_{41} in equations (3.15) can be replaced by L_4 to yield an equation in the unknown terms L_1, L_2, L_3, and L_4. This equation can be rearranged as

$$G_1 (L_1^4 L_2^4) + G_2 (L_1^4 L_2^2) + G_3 (L_3^2 L_1^2 L_2^2) + G_4 (L_1^2 L_2^2) + G_5 (L_3^2 L_2^2)$$

$$+ G_6 (L_2^2) + G_7 (L_3^2 L_1^2) + G_8 (L_1^2) + G_9 (L_3^4) + G_{10} (L_3^2) + G_{11} = 0 \qquad (4\text{-}1)$$

where

$$G_1 = 1$$

$$G_2 = 1$$

$$G_3 = -1$$

$$G_4 = -L_4^2 - a_{12}^2 - a_{34}^2$$

$$G_5 = a_{12}^2 - L_4^2$$

$$G_6 = L_4^2 a_{34}^2 - a_{34}^2 a_{12}^2$$

$$G_7 = -L_4^2 L_1^2$$

$$G_8 = - a_{34}^2 a_{12}^2 - a_{12}^2 L_4^2$$

$$G_9 = L_4^2$$

$$G_{10} = L_4^4 - a_{12}^2 L_4^2 - a_{34}^2 a_{12}^2 - a_{34}^2 L_4^2$$

$$G_{11} = -L_4^2 a_{34}^2 a_{12}^2 + a_{34}^4 a_{12}^2 - a_{12}^4 a_{34}^2 . \qquad (4\text{-}2)$$

The potential energy of the system can be evaluated as

$$U = \frac{1}{2} k_1 (L_1 - L_{01})^2 + \frac{1}{2} k_2 (L_2 - L_{02})^2 + \frac{1}{2} k_3 (L_3 - L_{03})^2 + \frac{1}{2} k_4 (L_4 - L_{04})^2 \qquad (4\text{-}3)$$

At equilibrium, the potential energy will be a minimum. This condition can be determined as the configuration of the mechanism whereby the derivative of the potential energy taken with respect to the lengths L_1, L_2, and L_3 all equal zero, i.e.

$$\frac{dU}{dL_1} = k_1 (L_1 - L_{01}) + k_4 (L_4 - L_{04})\frac{dL_4}{dL_1} = 0 \ , \tag{4-4}$$

$$\frac{dU}{dL_2} = k_2 (L_2 - L_{02}) + k_4 (L_4 - L_{04})\frac{dL_4}{dL_2} = 0 \ , \tag{4-5}$$

$$\frac{dU}{dL_3} = k_3 (L_3 - L_{03}) + k_4 (L_4 - L_{04})\frac{dL_4}{dL_3} = 0 \ . \tag{4-6}$$

The derivatives dL_4/dL_1, dL_4/dL_2, and dL_4/dL_3 are determined by implicit differentiation of Equation 4-1 as

$$\frac{dL_4}{dL_1} = \frac{-L_1[L_2^4 - L_2^2 a_{34}^2 + 2L_2^2 L_1^2 - L_2^2 L_3^2 - L_2^2 a_{12}^2 - L_2^2 L_4^2 + a_{12}^2 L_4^2 - L_4^2 L_3^2 - a_{34}^2 a_{12}^2 + L_3^2 a_{34}^2)]}{L_4 (L_2^2 a_{34}^2 - L_2^2 L_1^2 - L_2^2 L_3^2 + a_{12}^2 L_1^2 - L_3^2 L_1^2 - a_{34}^2 a_{12}^2 + 2L_4^2 L_3^2 + L_3^4 - L_3^2 a_{34}^2 - L_3^2 a_{12}^2} \tag{4-7}$$

$$\frac{dL_4}{dL_2} = \frac{-L_2[-2L_2^2 L_1^2 + a_{34}^2 L_1^2 - L_1^4 + L_3^2 L_1^2 + a_{12}^2 L_1^2 - L_4^2 a_{34}^2 + L_4^2 L_1^2 + a_{34}^2 a_{12}^2 + L_4^2 L_3^2 - L_3^2 a_{12}^2)]}{L_4 (L_2^2 a_{34}^2 - L_2^2 L_1^2 - L_2^2 L_3^2 + a_{12}^2 L_1^2 - L_3^2 L_1^2 - a_{34}^2 a_{12}^2 + 2L_4^2 L_3^2 + L_3^4 - L_3^2 a_{34}^2 - L_3^2 a_{12}^2)} \tag{4-8}$$

$$\frac{dL_4}{dL_3} = \frac{-L_3[-L_2^2 L_1^2 - L_2^2 L_4^2 - L_2^2 a_{12}^2 - L_4^2 L_1^2 + L_4^4 + 2L_4^2 L_3^2 - L_4^2 a_{34}^2 - a_{12}^2 L_4^2 - a_{34}^2 a_{12}^2 - a_{34}^2 L_1^2)]}{L_4 (L_2^2 a_{34}^2 - L_2^2 L_1^2 - L_2^2 L_3^2 + a_{12}^2 L_1^2 - L_3^2 L_1^2 - a_{34}^2 a_{12}^2 + 2L_4^2 L_3^2 + L_3^4 - L_3^2 a_{34}^2 - L_3^2 a_{12}^2)} \tag{4-9}$$

Substituting Equation 4-7 into Equation 4-4 and rearranging gives

$$D_1(L_1 L_2^4) + D_2(L_1^3 L_2^2) + D_3(L_1^2 L_2^2) + D_4(L_3^2 L_1 L_2^2) + D_5(L_1 L_2^2)$$

$$+ D_6(L_3^2 L_2^2) + D_7(L_2^2) + D_8(L_3^4) + D_9(L_3^2) + D_{10} + D_{11}(L_3^2 L_1^3) + D_{12}(L_1^3)$$

$$+ D_{13}(L_3^2 L_1^2) + D_{14}(L_1^2) + D_{15}(L_3^4 L_1) + D_{16}(L_3^2 L_1) + D_{17}(L_1) = 0 \tag{4-10}$$

where

$D_1 = -k_4 L_4 + k_4 L_{04} \ ,$

$D_2 = -2k_4 L_4 + 2k_4 L_{04} - k_1 L_4 \ ,$

$D_3 = k_1 L_4 L_{01} \ ,$

$D_4 = -k_1 L_4 - k_4 L_{04} + k_4 L_4 \ ,$

$D_5 = -k_4 L_{04} L_4^2 + k_4 L_4^3 + k_1 L_4 a_{34}^2 + k_4 L_4 a_{12}^2 - k_4 L_{04} a_{12}^2 + k_4 a_{34}^2 L_4 - k_4 L_{04} a_{34}^2 \ ,$

$D_6 = k_1 L_4 L_{01} \ ,$

$D_7 = -k_1 L_4 L_{01} a_{34}^2 \ ,$

$D_8 = k_1 L_4 L_{01}$,

$D_9 = k_1 L_4 L_{01} a_{12}{}^2 - 2k_1 L_4{}^3 L_{01} + k_1 L_4 L_{01} a_{34}{}^2$,

$D_{10} = k_1 L_4 L_{01} a_{34}{}^2 a_{12}{}^2$,

$D_{11} = k_1 L_4$,

$D_{12} = k_1 L_4 a_{12}{}^2$,

$D_{13} = k_1 L_4 L_{01}$,

$D_{14} = -k_1 L_4 L_{01} a_{12}{}^2$,

$D_{15} = k_1 L_4$,

$D_{16} = 2k_1 L_4{}^3 + k_4 L_4{}^3 + k_4 L_{04} a_{34}{}^2 - k_1 L_4 a_{12}{}^2 - k_4 a_{34}{}^2 L_4 + k_1 L_4 a_{34}{}^2 - k_4 L_{04} L_4{}^2$,

$D_{17} = -k_4 a_{12}{}^2 L_4{}^3 - k_1 L_4 a_{34}{}^2 a_{12}{}^2 + k_4 L_4 a_{34}{}^2 a_{12}{}^2 + k_4 L_{04} a_{12}{}^2 L_4{}^2 - k_4 L_{04} a_{34}{}^2 a_{12}$.   (4-11)

Substituting Equation 4-8 into Equation 4-5 and rearranging gives

$E_1(L_1{}^2 L_2{}^3) + E_2(L_3{}^2 L_2{}^3) + E_3(L_2{}^3) + E_4(L_1{}^2 L_2{}^2) + E_5(L_3{}^2 L_2{}^2) + E_6(L_2{}^2) + E_7(L_1{}^4 L_2)$

$+ E_8(L_3{}^2 L_2) + E_9(L_1{}^2 L_2) + E_{10}(L_3{}^4 L_2) + E_{11}(L_3{}^2 L_1{}^2 L_2) + E_{12}(L_2) + E_{13}(L_3{}^2)$

$+ E_{14}(L_1{}^2) + E_{15}(L_3{}^4) + E_{16} = 0$ (4-12)

where

$E_1 = -k_2 L_4 - 2k_4 L_4 + 2k_4 L_{04}$,

$E_2 = -k_2 L_4$,

$E_3 = k_2 L_4 a_{34}{}^2$,

$E_4 = k_2 L_4 L_{02}$,

$E_5 = k_2 L_4 L_{02}$,

$E_6 = -k_2 L_4 L_{02} a_{34}{}^2$,

$E_7 = k_4 L_{04} - k_4 L_4$,

$E_8 = k_4 L_4{}^3 - k_4 L_{04} L_4{}^2 + 2k_2 L_4{}^3 - k_2 L_4 a_{34}{}^2 - k_2 L_4 a_{12}{}^2 - k_4 L_4 a_{12}{}^2 + k_4 L_{04} a_{12}{}^2$,

$$E_9 = k_4L_4{}^3 + k_2L_4a_{12}{}^2 + k_4L_4a_{34}{}^2 + k_4L_4a_{12}{}^2 - k_4L_{04}a_{34}{}^2 - k_4L_{04}a_{12}{}^2 - k_4L_{04}L_4{}^2 \ ,$$

$$E_{10} = k_2L_4 \ ,$$

$$E_{11} = -k_2L_4 + k_4L_4 - k_4L_{04} \ ,$$

$$E_{12} = -k_4L_4{}^3a_{34}{}^2 + k_4L_4a_{34}{}^2a_{12}{}^2 - k_4L_{04}a_{34}{}^2a_{12}{}^2 - k_2L_4a_{34}{}^2a_{12}{}^2 + k_4L_{04}L_4{}^2a_{34}{}^2 \ ,$$

$$E_{13} = k_2L_4L_{02}a_{12}{}^2 - 2k_2L_4{}^3L_{02} + k_2L_4L_{02}a_{34}{}^2 + k_2L_4L_{02}L_1{}^2$$

$$E_{14} = -k_2L_4L_{02}a_{12}{}^2 \ ,$$

$$E_{15} = -k_2L_4L_{02} \ ,$$

$$E_{16} = k_2L_4L_{02}a_{34}{}^2a_{12}{}^2 \ . \tag{4-13}$$

Lastly, substituting Equation 4-9 into Equation 4-6 and rearranging gives

$$F_1(L_1{}^2L_2{}^2) + F_2(L_3L_1{}^2L_2{}^2) + F_3(L_3{}^3L_2{}^2) + F_4(L_3{}^2L_2{}^2) + F_5(L_3L_2{}^2) + F_6(L_2{}^2)$$

$$+ F_7(L_3{}^3L_1{}^2) + F_8(L_3\,L_1{}^2) + F_9(L_1{}^2) + F_{10}(L_3{}^5) + F_{11}(L_3{}^4) + F_{12}(L_3{}^3) + F_{13}(L_3{}^2)$$

$$+ F_{14}(L_3) + F_{15} = 0 \tag{4-14}$$

where

$$F_1 = k_3L_4L_{03} \ ,$$

$$F_2 = k_4L_4 - k_4L_{04} - k_3L_4 \ ,$$

$$F_3 = -k_3L_4 \ ,$$

$$F_4 = k_3L_4L_{03} \ ,$$

$$F_5 = k_3L_4a_{34}{}^2 - k_4L_4a_{12}{}^2 + k_4L_4{}^3 - k_4L_{04}L_4{}^2 + k_4L_{04}a_{12}{}^2 \ ,$$

$$F_6 = -k_3L_4L_{03}a_{34}{}^2 \ ,$$

$$F_7 = -k_3L_4 + k_3L_4L_{03} \ ,$$

$$F_8 = k_4L_4{}^3 + k_3L_4a_{12}{}^2 - k_4L_4a_{34}{}^2 - k_4L_{04}L_4{}^2 + k_4L_{04}a_{34}{}^2 \ ,$$

$$F_9 = -k_3L_4L_{03}a_{12}{}^2 \ ,$$

$$F_{10} = k_3L_4 \ ,$$

$$F_{11} = -k_3k_4L_{03} \ ,$$

$$F_{12} = 2k_3L_4{}^3 - k_3L_4a_{34}{}^2 - k_3L_4a_{12}{}^2 + 2k_4L_{04}L_4{}^2 - 2k_4L_4{}^3 \ ,$$

$$F_{13} = -2k_3L_4{}^3L_{03} + k_3L_4L_{03}a_{34}{}^2 + k_3L_4L_{03}a_{12}{}^2 \ ,$$

$$F_{14} = k_4 L_4^3 a_{34}^2 + k_4 a_{12}^2 L_4^3 + L_4 L_{04} L_4^4 + k_3 L_4 a_{34}^2 a_{12}^2 + k_4 L_4 a_{34}^2 a_{12}^2$$

$$- k_4 L_{04} L_4^2 a_{34}^2 - k_4 L_{04} a_{12}^2 L_4^2 - k_4 L_{04} a_{34}^2 a_{12}^2 - k_4 L_4^5,$$

$$F_{15} = k_3 L_4 L_{03} a_{34}^2 a_{12}^2. \tag{4-15}$$

4.4 Solution of Geometry and Energy Equations

The problem at hand is to determine sets of values for the parameters L_1, L_2, L_3, and L_4 that will simultaneously satisfy Equations (4-1), (4-10), (4-12), and (4-14). Sylvester's method is one approach for solving this set of equations for all possible solutions. The parameter L_4 has been embedded in the coefficients of Equations (4-1), (4-10), (4-12), and (4-14) and Table 4-1 shows the 37 products of L_1, L_2, and L_3 that exist in these four equations. The Table indicates which products that are multiplied by which coefficients for the four equations.

Table 4-1. Coefficient G, D, E, and F.

	$L_1^x\ L_2^y\ L_3^z$	G's	D's	E's	F's
1	$L_1^4 L_2^4$	G_1			
2	$L_1^4 L_2^2$	G_2			
3	$L_3^2 L_1^2 L_2^2$	G_3			
4	$L_1^2 L_2^2$	G_4	D_3	E_4	F_1
5	$L_3^2 L_2^2$	G_5	D_6	E_5	F_4
6	L_2^2	G_6	D_7	E_6	F_6
7	$L_3^2 L_1^2$	G_7	D_{13}		
8	L_1^2	G_8	D_{14}	E_{14}	F_9
9	L_3^4	G_9	D_8	E_{15}	F_{11}
10	L_3^2	G_{10}	D_9	E_{13}	F_{13}
11	1	G_{11}	D_{10}	E_{16}	F_{15}
12	$L_1 L_2^4$		D_1		
13	$L_1^3 L_2^2$		D_2		
14	$L_3^2 L_1 L_2^2$		D_4		
15	$L_1 L_2^2$		D_5		
16	$L_3^2 L_1^3$		D_{11}		
17	L_1^3		D_{12}		
18	$L_3^4 L_1$		D_{15}		
19	$L_3^2 L_1$		D_{16}		
20	L_1		D_{17}		
21	$L_1^2 L_2^3$			E_1	
22	$L_3^2 L_2^3$			E_2	
23	L_2^3			E_3	
24	$L_1^4 L_2$			E_7	
25	$L_3^2 L_2$			E_8	
26	$L_1^2 L_2$			E_9	

Table 4-1. Continued.

	$L_1{}^x L_2{}^y L_3{}^z$	G's	D's	E's	F's
27	$L_3{}^4 L_2$			E_{10}	
28	$L_3{}^2 L_1{}^2 L_2$			E_{11}	
29	L_2			E_{12}	
30	$L_3 L_1{}^2 L_2{}^2$				F_2
31	$L_3{}^3 L_2{}^2$				F_3
32	$L_3 L_2{}^2$				F_5
33	$L_3{}^3 L_1{}^2$				F_7
34	$L_3 L_1{}^2$				F_8
35	$L_3{}^5$				F_{10}
36	$L_3{}^3$				F_{12}
37	L_3				F_{14}

It is apparent from the complexity of the four equations that attempting to multiply the four equations by different products of L_1, L_2, and L_3 in order to obtain a set of "homogeneous" equations where the number of equations equals the number of unknowns is impractical. For this reason, a numerical case will be considered and this case will be solved using the continuation approach.

4.5 Numerical Case Study

The following parameters were selected as a numerical example:

Strut lengths: $a_{12} = 14$ in., $a_{34} = 12$ in.

Spring free lengths and spring constants:

$L_{01} = 8$ in. $\qquad$ $k_1 = 1$ lbf/in.

$L_{02} = 2.68659245$ in. $\quad$ $k_2 = 2.0$ lbf/in.

$L_{03} = 3.46513678$ in. $\quad$ $k_3 = 2.5$ lbf/in.

$L_{04} = 7.3082878$ in. $\qquad$ $k_4 = 1.5$ lbf/in.

Equations (4-1), (4-10), (4-12), and (4-14) can now be expressed respectively as

$$L_2{}^4 L_1{}^2 + (L_1{}^4 - L_3{}^2 L_1{}^2 - 440.0\ L_1{}^2 + 96.0\ L_3{}^2 - 13824.0)\ L_2{}^2 - 8424.0\ L_1{}^2 - 8624.0\ L_1{}^2 +$$

$$44.0\ L_3{}^2 L_1{}^2 + 6773760.0 - 52224.0\ L_3{}^2 + L_3{}^4 = 0 \qquad\qquad (4\text{-}16)$$

$- 4.03756830 \, L_2{}^4 \, L_1 + (-11520.0 - 5.96243170 \, L_3{}^2 \, L_1 + 80.0 \, L_1{}^2 - 18.07513660.0 \, L_1{}^3 +$

$3216.53005200 \, L_1 + 80.0 \, L_3{}^2) \, L_2{}^2 + 11200.0 \, L_3{}^2 + 1960.0 \, L_1{}^3 - 10 \, L_3{}^2 \, L_1{}^3 -$

$1577.65300520 \, L_3{}^2 \, L_1 + 10.0 \, L_1 \, L_3{}^4 - 80.0 \, L_3{}^4 + 0.2257920 \, 10^7 - 15680.0 \, L_1{}^2 -$

$247420.01098080 \, L_1 + 80. \, L_3{}^2 \, L_1{}^2 = 0$

$$(4\text{-}17)$$

$(-28.07513660 \, L_1{}^2 + 2880.0 - 20.0 \, L_3{}^2) \, L_2{}^3 + (-7737.38625600 + 53.73184900 \, L_3{}^2 +$

$53.73184900 \, L_1{}^2) \, L_2{}^2 + (-3187.60655680 \, L_3{}^2 + 5696.53005200 \, L_1{}^2 - 15.96243170 \, L_3{}^2$

$L_1{}^2 + 20.0 \, L_3{}^4 - 4.03756830 \, L_1{}^4 - 508664.65582080) \, L_2 + 7522.45886000 \, L_3{}^2 +$

$0.151652770617600 \, 10^7 - 53.73184900 \, L_3{}^4 - 10531.44240400 \, L_1{}^2 + 53.73184900 \, L_3{}^2$

$L_1{}^2 = 0$

$$(4\text{-}18)$$

$- 494742.03310080 \, L_3 + 12127.978730000 \, L_3{}^2 - 16979.170222000 \, L_1{}^2 + 86.628419500$

$L_3{}^2 \, L_1{}^2 - 86.628419500 \, L_3{}^4 - 4307.51366000 \, L_3{}^3 + 4722.34699480 \, L_3 \, L_1{}^2 + 25.0 \, L_3{}^5 -$

$25.0 \, L_3{}^3 \, L_1{}^2 + (-20.96243170 \, L_3 \, L_1{}^2 + 86.628419500 \, L_1{}^2 + 3212.39344320 \, L_3 - 25.0 \, L_3{}^3$

$- 12474.492408000 + 86.628419500 \, L_3{}^2) \, L_2{}^2 + 0.2445000511968000 \, 10^7 = 0$

$$(4\text{-}19)$$

The Polynomial Continuation Method was used to find simultaneous roots for the four equations. Eighteen real solutions and 490 complex solutions were found. All the real and complex solutions were shown to satisfy Equations (4-16) through (4-19) with neglible residuals. The real solutions are listed in Table 4-2.

Table 4-2. Eighteen real solutions. (Solutions that are geometrically impossible are shaded.)

	L_1, in.	L_2, in.	L_3, in.	L_4, in.
1	19.028528	3.624423	3.108440	4.306682
2	13.000001	7.999996	7.016593	9.999998
3	11.632986	-4.613617	9.950038	12.269992

Table 4-2. Continued.

	L_1, in.	L_2, in.	L_3, in.	L_4, in.
4	3.141567	2.156404	5.241983	18192258
5	-7.985150	10.049547	-6.186358	14.286022
6	-0.398136	4.497517	10.533965	15.669134
7	-6.868035	1.105366	3.486019	20.971818
8	4.200027	-3.538652	10.462012	15.411513
9	19.507910	4.034579	3.169571	5.347832
10	-16.748403	8.590494	2.764444	2.329743
11	4.520731	0.603385	-7.518882	18.852623
12	4.077402	4.708715	9.316941	16.646605
13	13.473294	11.823371	-2.235364	7.821817
14	22.800126	6.225147	13.541713	-5.861200
15	14.616270	-9.580282	-4.547034	8.863331
16	20.778750	-0.084697	4.369638	6.733159
17	-12.774217	12.496006	-1.650677	8.309365
18	3.806954	2.744884	8.098941	17.877213

Cases 1, 4, 7, 9, 10, 11, 14, 16, and 18 are not physically realizable. For example, L_1 in case 1 is greater than sum of a_{12} and L_4, and hence point 2 cannot be constructed.

The solutions in Table 4-2 can correspond to either maximum or minimum potential energy configurations. The second derivative of the potential energy with respect to the variables L_1, L_2, and L_3 are now evaluated to identity the minimal potential energy cases.

Derivatives of Equations (4-4) through (4-6) with respect to L_1, L_2, and L_3, respectively can be written as

$$\frac{d^2U}{dL_1^{\,2}} = k_1 + k_4(\frac{dL_4}{dL_1})^2 + k_4(L_4 - L_{04})\frac{d^2L_4}{dL^2_1} \ ,$$

$$\frac{d^2U}{dL_2^{\,2}} = k_2 + k_4(\frac{dL_4}{dL_2})^2 + k_4(L_4 - L_{04})\frac{d^2L_4}{dL^2_2} \ ,$$

$$\frac{d^2U}{dL_3^{\,2}} = k_3 + k_4(\frac{dL_4}{dL_3})^2 + k_4(L_4 - L_{04})\frac{d^2L_4}{dL^2_3} \ .$$

(4-20)

Table 4-3 shows the value of the potential energy and the second derivative for each of the remaining nine cases. It is seen that cases 2, 3, 13, 15, and 17 have all positive second derivative values and represent possible equilibrium solutions. Each of these solutions was analyzed to determine if it truly corresponds to an equilibrium configuration and only cases 2, 3, 13, and 15 were indeed in equilibrium.

Table 4-3. Identification of minimum potential energy state. (Solutions that do not have all positive second derivatives are shaded.)

Case	L_1, in	L_2, in	L_3, in	L_4, in	U, in-lbf	$\dfrac{\partial^2 U}{\partial L_1^{\,2}}$ lbf/in	$\dfrac{\partial^2 U}{\partial L_2^{\,2}}$ lbf/in	$\dfrac{\partial^2 U}{\partial L_3^{\,2}}$ lbf/in
2	13.000	7.999	7.016	9.999	61.932	1.146	3.052	4.569
3	11.632	-4.613	9.950	12.269	130.923	0.740	0.566	4.065
5	-7.985	10.049	-6.186	14.286	334.931	-0.470	1.241	1.594
6	-0.398	4.497	10.533	15.669	153.432	3.597	1.889	-0.750
8	4.200	-3.538	10.462	15.411	156.415	0.659	-16.134	-2494.334
12	4.077	4.708	9.316	16.646	119.990	-0.221	-15.521	-379.009
13	13.473	11.823	-2.235	7.821	139.276	4.967	51.239	56.665
15	14.616	-9.580	-4.547	8.863	25.420	2.405	25.608	32.573
17	-12.774	12.496	-1.650	8.309	345.474	10.529	50.466	39.862

Figure 4-2 shows the four feasible solutions that are in stable equilibrium drawn to scale. Dashed lines are used to identify the springs that have negative spring lengths. It must be noted that when a spring has a negative spring length, the two points at the end of the spring want to move towards one another as the spring attempts to return to its free length.
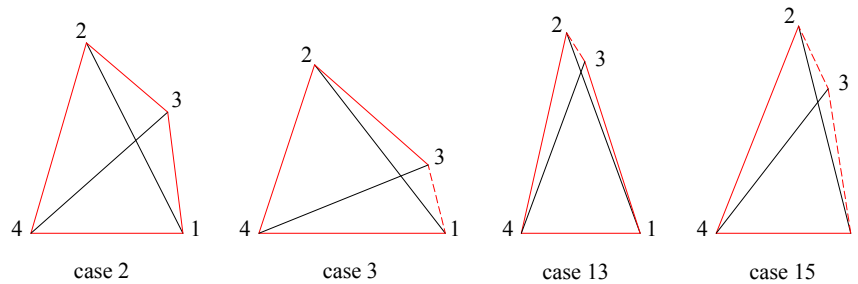

Figure 4-2. Four equilibrium solutions

A force analysis was conducted for each of the four cases and these cases were found to be in static equilibrium. Table 4-4 shows the calculated force in each of the four springs and the two struts for the equilibrium cases. Negative values in the table correspond to the springs which have a negative spring length.

Table 4-4. Calculated forces in struts, and elastic ties.

Case	Force in Spring L_1, lbf	Force in Spring L_2, lbf	Force in Spring L_3, lbf	Force in Spring L_4, lbf	Force in Strut a_{12}, lbf	Force in Strut a_{34}, lbf
2	5.000	10.625	8.877	4.036	11.828	7.220
3	3.632	- 14.599	16.212	7.441	17.942	9.253
13	5.473	18.273	- 14.250	0.769	18.5294	5.8228
15	6.616	- 24.533	- 20.030	2.332	25.005	7.6394

4.6 Conclusions

The problem of determining the equilibrium configurations of a four-spring planar tensegrity system was formulated by developing four equations in terms of the four spring lengths. It was not practical to use Sylvester's method to solve this set of equations. Rather, the continuation method was used for a particular numerical example and four equilibrium configurations were obtained. Cases may exist that would yield more than four realizable configurations, and this could be investigated in future work. The objective here, however, was to determine if it was tractable at all to solve the equilibrium problem for multiple (if not all) solutions, either symbolically or numerically, and this goal was met.

CHAPTER 5
FUTURE WORK, SUMMARY, AND CONCLUSION

5.1 Future Work

5.1.1 The 3D Tensegrity Platform

As part of future suggested research work, an introduction to the position analysis

of a general three dimensional parallel platform device consisting of 3 struts, 3 springs,

and 6 ties (Figure 5-1) is presented here. The problem statement is given as follows:

Given: length of non-compliant ties, struts, and free lengths and spring constants

of the compliant ties.

Find: final length of compliant ties at all equilibrium configurations.
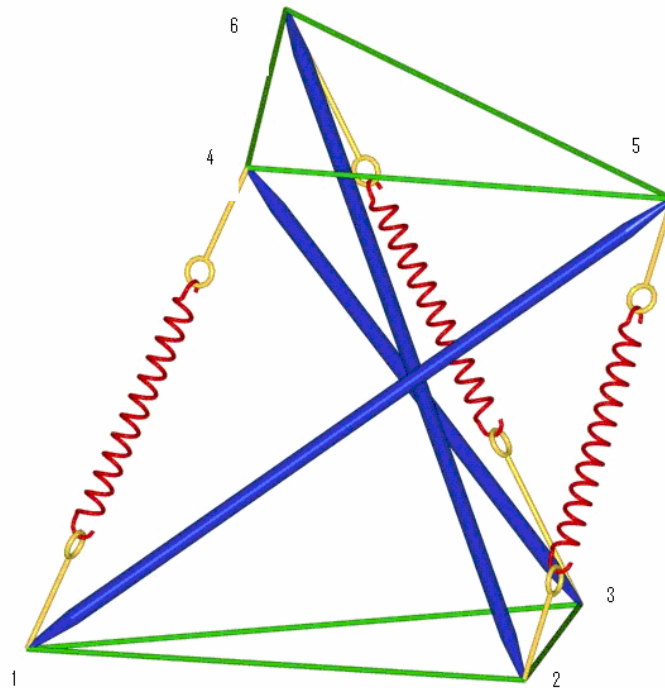
Figure 5-1. Three-dimensional parallel platform of 3 struts, 3 springs, and 6 ties.

Three approaches to this problem are presented here:

1.   Minimum potential energy analysis.
2.   Force balance analysis.
3.   Linear dependence of 6 connector lines.

The unexpected complexity, i.e. high degree polynomials, encountered in the planar analysis may indicate that the analysis of the general three strut tensegrity system will be very complex.

This is a three degree of freedom system. One approach is to define the system by six angles, subject to three constraints. That is, the coordinates of the tip of each strut (points 4, 5, and 6) can be defined by angles α_i and β_i, i = 1..3, which define the orientation of each of the three struts relative to the base triangle. The three constraints are that the distances between points 4, 5, and 6 are given. The potential energy can be written in terms of these six angles and three equations can be generated by equating the partial derivative of the potential energy with respect to three of the angles to zero. These three equations, together with the three distance constraint equations would give a set of six equations in the six angles.

In the second approach, force balance equations at points 4, 5, and 6 could be written as functions of the lengths of the three springs. This is complicated in that the positions of points 4, 5, and 6 would have to be determined from a forward analysis of the device, i.e. the device is equivalent to a 3-3 parallel mechanism, whose forward analysis was solved by Griffis and Duffy [27]. The solution of the forward analysis of the 3-3 parallel mechanism requires rooting a fourth degree polynomial in the square of one parameter, resulting in eight possible configurations for the device. The nature of the

forward analysis makes it infeasible to obtain a symbolic solution to the equilibrium configuration problem. A numerical solution may be possible.

The third approach utilizes the fact that if the device is in equilibrium with no external forces applied, then the Plücker coordinates of the lines along the three struts and the three springs must be linearly dependent. For this solution, it is then necessary to obtain the Plücker coordinates of these lines as functions of the three spring lengths. This again requires a forward analysis of the 3-3 parallel mechanism. Further, only a single equation results from equating the determinant of the 6×6 matrix of line coordinates to zero. This leads one to believe that the fact that the lines must be linearly dependent is a necessary, but not sufficient condition for equilibrium. Further analysis is necessary.

5.1.2 The 2D Tensegrity structures

Cases of two dimensional tensegrity structures proved to be mathematically challenging in spite of their simple configurations. Different methods were tried to solve these problems. It was expected the final closed form solution be a simple mathematical expression. In each case the intermediate mathematical expressions become complicated to solve and the relevant matrices become large to manipulate. Other mathematical procedures might exist to solve these problems with less complexity.

5.2: Summary and Conclusion

In this research, all possible combinations of two dimensional tensegrity structures consisting of two struts, springs, and non-compliant members were analyzed. A closed form solution could only be obtained for case of 2 spring, 2 struts, and 2 non-compliant members. Finding a simple solution approach for a closed form solution to the other two dimensional tensegrity structures consisting of either 3 or 4 springs was not possible due

to the complexity of mathematical solutions. A numerical solution using the Continuity Method was used to solve individual numerical examples.

The contribution of this research was to set a basic mathematical foundation for the closed form solutions of tensegrity structures and also to present numerical approaches where a closed form solution could not be obtained. The contribution of this research leads to a better understanding of the complexity of these devices and hopes to lay a ground work for future analyses of more complex systems.

The following coefficients belong to the numerical example of a Maple sessions programming.

$$(p_1 \, c_1^2 + p_2 \, c_1 + p_3) \, c_4^3 + (p_4 \, c_1^3 + p_5 \, c_1^2 + p_6 \, c_1 + p_7) \, c_4^2 + (p_8 \, c_1^4 + p_9 \, c_1^3 + p_{10} \, c_1^2 +$$
$$p_{11} \, c_1 + p_{12}) \, c_4 + (p_{13} \, c_1^4 + p_{14} \, c_1^3 + p_{15} \, c_1^2 + p_{16} \, c_1 + p_{17}) = 0 \qquad (3\text{-}65)$$

$$(q_1 \, c_1 + q_2) \, c_4^4 + (q_3 \, c_1^2 + q_4 \, c_1 + q_5) \, c_4^3 + (q_6 \, c_1^3 + q_7 \, c_1^2 + q_8 \, c_1 + q_9) \, c_4^2 + (q_{10} \, c_1^3 + q_{11}$$
$$c_1^2 + q_{12} \, c_1 + q_{13}) \, c_4 + (q_{14} \, c_1^3 + q_{15} \, c_1^2 + q_{16} \, c_1 + q_{17}) = 0 \qquad (3\text{-}66)$$

$$(r_1 \, c_1 + r_2) \, c_4^2 + (r_3 \, c_1^2 + r_4 \, c_1 + r_5) \, c_4 + (r_6 \, c_1^2 + r_7 \, c_1 + r_8) = 0 \qquad (3\text{-}69)$$

```
> #----------------------------------------------------------
> # Numerical example
```

$a12 := 14 \quad a34 := 12 \quad a41 := 10$

$L01 := 8 \quad k1 := 1$

$L02 := 2 \quad k2 := 2.687$

$L03 := 2.5 \quad k3 := 3.465$

$p1 := 0.5773663961 \times 10^{15} \, L3^2 - 0.2080248063 \times 10^{16} \, L3 + 0.1873780685 \times 10^{16}$

$p2 := 0.1110100300 \times 10^{16} \, L3^2 - 0.4198962128 \times 10^{16} \, L3 + 0.3961707733 \times 10^{16}$

$p3 := 0.5335964670 \times 10^{15} \, L3^2 - 0.2114119475 \times 10^{16} \, L3 + 0.2094045517 \times 10^{16}$

$p4 := 0.4853912146 \times 10^{15} \, L3^2 - 0.2087909024 \times 10^{16} \, L3 + 0.2186077465 \times 10^{16}$

$p5 := -0.1733540051 \ 10^{13} \ L3^4 + 0.7456817937 \ 10^{13} \ L3^3 + 0.2423387462 \ 10^{16} \ L3^2$
$\qquad - 0.9462608682 \ 10^{16} \ L3 + 0.9212755033 \ 10^{16}$

$p6 := -0.3864997467 \ 10^{13} \ L3^4 + 0.1452389258 \ 10^{14} \ L3^3 + 0.3406073884 \ 10^{16} \ L3^2$
$\qquad - 0.1266504024 \ 10^{17} \ L3 + 0.1176377359 \ 10^{17}$

$p7 := -0.2113487353 \ 10^{13} \ L3^4 + 0.7162795261 \ 10^{13} \ L3^3 + 0.1466145513 \ 10^{16} \ L3^2$
$\qquad - 0.5300984714 \ 10^{16} \ L3 + 0.4722921577 \ 10^{16}$

$p8 := 0.1020169484 \ 10^{15} \ L3^2 - 0.5100847420 \ 10^{15} \ L3 + 0.6376059274 \ 10^{15}$

$p9 := -0.7286924884 \ 10^{12} \ L3^4 + 0.3643462442 \ 10^{13} \ L3^3 + 0.1068305616 \ 10^{16} \ L3^2$
$\qquad - 0.4737062676 \ 10^{16} \ L3 + 0.5137282043 \ 10^{16}$

$p10 := 0.1301236586 \ 10^{10} \ L3^6 - 0.6506182936 \ 10^{10} \ L3^5 - 0.4747632005 \ 10^{13} \ L3^4$
$\qquad + 0.2052962276 \ 10^{14} \ L3^3 + 0.3082075504 \ 10^{16} \ L3^2 - 0.1236996613 \ 10^{17} \ L3$
$\qquad + 0.1219206631 \ 10^{17}$

$p11 := 0.3300435984 \ 10^{10} \ L3^6 - 0.1289836349 \ 10^{11} \ L3^5 - 0.7989672476 \ 10^{13} \ L3^4$
$\qquad + 0.2941695331 \ 10^{14} \ L3^3 + 0.3455740168 \ 10^{16} \ L3^2 - 0.1248534533 \ 10^{17} \ L3$
$\qquad + 0.1113665191 \ 10^{17}$

$p12 := 0.2092793462 \ 10^{10} \ L3^6 - 0.5893635686 \ 10^{10} \ L3^5 - 0.3983027155 \ 10^{13} \ L3^4$
$\qquad + 0.1246857458 \ 10^{14} \ L3^3 + 0.1339547765 \ 10^{16} \ L3^2 - 0.4345164946 \ 10^{16} \ L3$
$\qquad + 0.3440522646 \ 10^{16}$

$p13 := 0.1037172309 \ 10^{15} \ L3^2 - 0.5185861544 \ 10^{15} \ L3 + 0.6482326928 \ 10^{15}$

$p14 := -0.7408373632 \ 10^{12} \ L3^4 + 0.3704186816 \ 10^{13} \ L3^3 + 0.5844049572 \ 10^{15} \ L3^2$
$\qquad - 0.2657927757 \ 10^{16} \ L3 + 0.2963349453 \ 10^{16}$

$p15 := 0.1322923863 \ 10^{10} \ L3^6 - 0.6614619316 \ 10^{10} \ L3^5 - 0.3034952946 \ 10^{13} \ L3^4$
$\qquad + 0.1316433327 \ 10^{14} \ L3^3 + 0.1235296500 \ 10^{16} \ L3^2 - 0.4981478398 \ 10^{16} \ L3$
$\qquad + 0.4841901337 \ 10^{16}$

$p16 := 0.3355443251 \ 10^{10} \ L3^6 - 0.1311333621 \ 10^{11} \ L3^5 - 0.4128749122 \ 10^{13} \ L3^4$
$\qquad + 0.1489517912 \ 10^{14} \ L3^3 + 0.1159476235 \ 10^{16} \ L3^2 - 0.4015136920 \ 10^{16} \ L3$
$\qquad + 0.3326208569 \ 10^{16}$

$p17 := 0.2121019430 \ 10^{10} \ L3^6 - 0.5991862948 \ 10^{10} \ L3^5 - 0.1862278592 \ 10^{13} \ L3^4$
$\qquad + 0.5272839394 \ 10^{13} \ L3^3 + 0.4069170754 \ 10^{15} \ L3^2 - 0.1160024667 \ 10^{16} \ L3$
$\qquad + 0.8167029969 \ 10^{15}$

$$q1 := 0.8744309862 \; 10^{14} \, L3^2 - 0.4372154932 \; 10^{15} \, L3 + 0.5465193664 \; 10^{15}$$

$$q2 := 0.9243984712 \; 10^{14} \, L3^2 - 0.4621992356 \; 10^{15} \, L3 + 0.5777490445 \; 10^{15}$$

$$q3 := 0.4669520350 \; 10^{15} \, L3^2 - 0.2187549572 \; 10^{16} \, L3 + 0.2550423709 \; 10^{16}$$

$$q4 := -0.7286924880 \; 10^{12} \, L3^4 + 0.3643462442 \; 10^{13} \, L3^3 + 0.9975040455 \; 10^{15} \, L3^2 - 0.4749518798 \; 10^{16} \, L3 + 0.5610932160 \; 10^{16}$$

$$q5 := -0.7703320590 \; 10^{12} \, L3^4 + 0.3851660295 \; 10^{13} \, L3^3 + 0.5326615535 \; 10^{15} \, L3^2 - 0.2576221616 \; 10^{16} \, L3 + 0.3081328237 \; 10^{16}$$

$$q6 := 0.6233888280 \; 10^{15} \, L3^2 - 0.2723886871 \; 10^{16} \, L3 + 0.2975494328 \; 10^{16}$$

$$q7 := -0.1945633479 \; 10^{13} \, L3^4 + 0.9114789880 \; 10^{13} \, L3^3 + 0.2467565040 \; 10^{16} \, L3^2 - 0.1115537420 \; 10^{17} \, L3 + 0.1253958324 \; 10^{17}$$

$$q8 := 0.1518109351 \; 10^{10} \, L3^6 - 0.7590546756 \; 10^{10} \, L3^5 - 0.4948249499 \; 10^{13} \, L3^4 + 0.2326400716 \; 10^{14} \, L3^3 + 0.2978451536 \; 10^{16} \, L3^2 - 0.1366694272 \; 10^{17} \, L3 + 0.1548714436 \; 10^{17}$$

$$q9 := 0.1604858457 \; 10^{10} \, L3^6 - 0.8024292286 \; 10^{10} \, L3^5 - 0.3056661932 \; 10^{13} \, L3^4 + 0.1440713627 \; 10^{14} \, L3^3 + 0.1127231047 \; 10^{16} \, L3^2 - 0.5199241185 \; 10^{16} \, L3 + 0.5873781950 \; 10^{16}$$

$$q10 := 0.1238733084 \; 10^{16} \, L3^2 - 0.5475596549 \; 10^{16} \, L3 + 0.6050171798 \; 10^{16}$$

$$q11 := -0.4022366239 \; 10^{13} \, L3^4 + 0.1777220459 \; 10^{14} \, L3^3 + 0.3554116422 \; 10^{16} \, L3^2 - 0.1568854809 \; 10^{17} \, L3 + 0.1720685863 \; 10^{17}$$

$$q12 := 0.3260393580 \; 10^{10} \, L3^6 - 0.1447643958 \; 10^{11} \, L3^5 - 0.7932329962 \; 10^{13} \, L3^4 + 0.3478741954 \; 10^{14} \, L3^3 + 0.3352073482 \; 10^{16} \, L3^2 - 0.1470298991 \; 10^{17} \, L3 + 0.1588185279 \; 10^{17}$$

$$q13 := 0.3446701784 \; 10^{10} \, L3^6 - 0.1530366470 \; 10^{11} \, L3^5 - 0.3895030685 \; 10^{13} \, L3^4 + 0.1691392672 \; 10^{14} \, L3^3 + 0.1035832980 \; 10^{16} \, L3^2 - 0.4479338270 \; 10^{16} \, L3 + 0.4707584806 \; 10^{16}$$

$$q14 := 0.6153702088 \; 10^{15} \, L3^2 - 0.2751416753 \; 10^{16} \, L3 + 0.3075503998 \; 10^{16}$$

$$q15 := -0.2075805867 \; 10^{13} \, L3^4 + 0.8667876209 \; 10^{13} \, L3^3 + 0.1553312052 \; 10^{16} \, L3^2 - 0.6722506098 \; 10^{16} \, L3 + 0.7212669096 \; 10^{16}$$

$$q16 := 0.1750560045 \; 10^{10} \, L3^6 - 0.6792486624 \; 10^{10} \, L3^5 - 0.3717087746 \; 10^{13} \, L3^4 + 0.1514057165 \; 10^{14} \, L3^3 + 0.1284193314 \; 10^{16} \, L3^2 - 0.5346826116 \; 10^{16} \, L3 + 0.5463422536 \; 10^{16}$$

$$q17 := -0.7180628716 \ 10^{10} \ L3^5 - 0.1603893518 \ 10^{13} \ L3^4 + 0.6318953271 \ 10^{13} \ L3^3$$
$$+ \ 0.1830521647 \ 10^{10} \ L3^6 - 0.1390169719 \ 10^{16} \ L3 + 0.1348526898 \ 10^{16}$$
$$+ \ 0.3482593233 \ 10^{15} \ L3^2$$

$$r1 := 8 \ a12 \ a41 \ a34^2$$

$$r2 := 4 \ a41^2 \ a34^2 + 4 \ a34^2 \ a12^2$$

$$r3 := 8 \ a34 \ a41 \ a12^2$$

$$r4 := 12 \ a12 \ a41^2 \ a34 - 4 \ a12 \ a34 \ L3^2 + 4 \ a12 \ a34^3 + 4 \ a12^3 \ a34$$

$$r5 := -4 \ a34 \ a41 \ L3^2 + 4 \ a41^3 \ a34 + 4 \ a34 \ a41 \ a12^2 + 4 \ a34^3 \ a41$$

$$r6 := 4 \ a41^2 \ a12^2 + 4 \ a34^2 \ a12^2$$

$$r7 := 4 \ a12^3 \ a41 - 4 \ a12 \ a41 \ L3^2 + 4 \ a12 \ a41 \ a34^2 + 4 \ a12 \ a41^3$$

$$r8 := L3^4 + (-2 \ a34^2 - 2 \ a41^2 - 2 \ a12^2) \ L3^2 + a41^4 + a34^4 + a12^4 + 2 \ a41^2 \ a34^2$$
$$+ \ 2 \ a41^2 \ a12^2 - 2 \ a34^2 \ a12^2$$

The following portion of Maple program presents the three equations used in continuation method.

$$(3\text{-}71) = \texttt{eqn1\_for\_cont}$$

$$(3\text{-}72) = \texttt{eqn3\_16\_sq}$$

$$(3\text{-}73) = \texttt{eqn3\_17\_sq}$$

```
> # 3 springs cosine of theta1, cosine of theta2 and L23, Continuation method
#++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
> a41:= 10 ;   a12:= 14.0 ;   a34:= 12 ;
```
$$a41 := 10$$
$$a12 := 14.0$$
$$a34 := 12$$

```
> k24:= 1 ;   k31:= 2.0 ;   k23:= 2.5 ;
```
$$k24 := 1$$
$$k31 := 2.0$$

$$k23 := 2.5$$

```
>
> L024:= 8.0;   L031:= 2.686592430;   L023:= 3.465136806;
```

$$L024 := 8.0$$

$$L031 := 2.686592430$$

$$L023 := 3.465136806$$

```
>  eqn1_for_cont:= eqn1_for_cont ;
```

$$eqn1\_for\_cont := 61600.000\ c1 + 42624.00\ c4^2 - 168.0\ c1\ c4\ L23^2 + \frac{1}{4}L23^4$$

$$- 220.0000000\ L23^2 + 47824.00\ c1^2 + 52800.00\ c4 - 140.0\ c1\ L23^2$$

$$+ 47040.00\ c1^2\ c4 + 40320.0\ c1\ c4^2 + 107520.000\ c1\ c4 + 20176.00000$$

$$- 120\ c4\ L23^2$$

```
>
> eqn3_16_sq:= eqn3_16_sq ;
```

$$eqn3\_16\_sq := 0.2094210873\ 10^{16}\ c4^3 + 0.1119744000\ 10^{10}\ L23^6\ c4$$

$$+ 0.1131754495\ 10^{10}\ L23^6 - 0.4383037685\ 10^{10}\ L23^5 + 0.3326471226\ 10^{16}\ c1$$

$$+ 0.4723294525\ 10^{16}\ c4^2 + 0.5399138304\ 10^{14}\ L23^2\ c1^4$$

$$+ 0.1817487566\ 10^{16}\ c1\ c4\ L23^2 - 0.3619570961\ 10^{16}\ L23\ c1^2$$

$$+ 0.3038915898\ 10^{15}\ L23^2\ c1^3 - 0.4221575075\ 10^{13}\ L23^4\ c1\ c4$$

$$- 0.9917251153\ 10^{12}\ L23^4 + 0.2153950907\ 10^{15}\ L23^2 + 0.4842283683\ 10^{16}\ c1^2$$

$$- 0.8485560960\ 10^{15}\ L23 + 0.3440794329\ 10^{16}\ c4$$

$$+ 0.1488688451\ 10^{14}\ L23^3\ c1^2\ c4 - 0.2488945515\ 10^{13}\ L23^4\ c1^2\ c4$$

$$+ 0.6376562766\ 10^{15}\ c1^4\ c4 + 0.1275251673\ 10^{16}\ L23^2\ c1^2\ c4^2$$

$$+ 0.1614293490\ 10^{16}\ L23^2\ c1^2\ c4 + 0.1795299633\ 10^{16}\ L23^2\ c4^2\ c1$$

$$- 0.3793305600\ 10^{12}\ L23^4\ c1^3\ c4 - 0.4694401018\ 10^{10}\ L23^5\ c1^2\ c4$$

$$+ 0.6773760000\ 10^{9}\ L23^6\ c1^2\ c4 - 0.2043740160\ 10^{13}\ L23^4\ c1\ c4^2$$

$$- 0.9388802036\ 10^{10}\ L23^5\ c1\ c4 - 0.1535416328\ 10^{16}\ L23\ c4^3$$

$$+ 0.1741824000\ 10^{10}\ L23^6\ c1\ c4 + 0.5580683989\ 10^{15}\ L23^2\ c1^3\ c4$$

$$+ 0.2141076067\ 10^{14}\ L23^3\ c1\ c4 + 0.2628864571\ 10^{13}\ L23^3\ c1^3\ c4$$

$$+ 0.9213482524\ 10^{16}\ c1^2\ c4^2 + 0.2186250090\ 10^{16}\ c1^3\ c4^2$$

$$+ 0.6089439462\ 10^{15}\ c1\ L23^2 - 0.9103933440\ 10^{12}\ L23^4\ c1^2\ c4^2$$

$$+ 0.5137687713\ 10^{16}\ c1^3\ c4 + 0.1055837852\ 10^{14}\ L23^3\ c1\ c4^2$$

$$+ 0.6445019392\ 10^{15}\ L23^2\ c1^2 + 0.5407949973\ 10^{13}\ L23^3\ c1^2\ c4^2$$

$$+ 0.3857073162\ 10^{13}\ L23^3 + 0.1873928649\ 10^{16}\ c1^2\ c4^3$$

$$+ 0.3962020571 \ 10^{16} \ c1 \ c4^3 - 0.3741750572 \ 10^{15} \ L23 \ c1^4$$

$$+ 0.5310627840 \ 10^{14} \ L23^2 \ c1^4 \ c4 - 0.4311184608 \ 10^{10} \ L23^5 \ c4$$

$$+ 0.2963583455 \ 10^{16} \ c1^3 + 0.2814309827 \ 10^{15} \ L23^2 \ c4^3$$

$$+ 0.2549101363 \ 10^{15} \ L23^2 \ c1^3 \ c4^2 + 0.1219302908 \ 10^{17} \ c1^2 \ c4$$

$$+ 0.5868135383 \ 10^{15} \ L23^2 \ c4^3 \ c1 - 0.2119474277 \ 10^{13} \ L23^4 \ c4$$

$$- 0.3053174043 \ 10^{16} \ L23 \ c4^3 \ c1 - 0.3856527360 \ 10^{12} \ L23^4 \ c1^3$$

$$- 0.9545282070 \ 10^{10} \ L23^5 \ c1 - 0.1122729984 \ 10^{13} \ L23^4 \ c4^2$$

$$+ 0.6482838809 \ 10^{15} \ c1^4 + 0.9545282070 \ 10^{13} \ L23^3 \ c1^2$$

$$+ 0.7738436783 \ 10^{15} \ L23^2 \ c4^2 + 0.1176470253 \ 10^{17} \ c1 \ c4^2$$

$$- 0.4772641035 \ 10^{10} \ L23^5 \ c1^2 + 0.1113753133 \ 10^{17} \ c1 \ c4$$

$$+ 0.6886656000 \ 10^{9} \ L23^6 \ c1^2 + 0.1085434933 \ 10^{14} \ L23^3 \ c1$$

$$- 0.1924328866 \ 10^{16} \ L23 \ c1^3 - 0.2180321636 \ 10^{13} \ L23^4 \ c1$$

$$+ 0.5224006096 \ 10^{13} \ L23^3 \ c4^2 - 0.2927947095 \ 10^{16} \ L23 \ c1$$

$$- 0.1514225992 \ 10^{16} \ L23 \ c1^2 \ c4^3 + 0.1770854400 \ 10^{10} \ L23^6 \ c1$$

$$- 0.6875307066 \ 10^{16} \ L23 \ c1^2 \ c4^2 - 0.9212468531 \ 10^{16} \ L23 \ c1 \ c4^2$$

$$+ 0.3058921636 \ 10^{15} \ L23^2 \ c1^2 \ c4^3 - 0.3171517401 \ 10^{16} \ L23 \ c4$$

$$- 0.3859569282 \ 10^{16} \ L23 \ c4^2 + 0.9104915321 \ 10^{13} \ L23^3 \ c4$$

$$- 0.9090163022 \ 10^{16} \ L23 \ c1 \ c4 + 0.8167674885 \ 10^{15}$$

$$- 0.1589435265 \ 10^{13} \ L23^4 \ c1^2 + 0.7079662564 \ 10^{15} \ c4 \ L23^2$$

$$- 0.3432245583 \ 10^{16} \ L23 \ c1^3 \ c4 - 0.8985008439 \ 10^{16} \ L23 \ c1^2 \ c4$$

$$+ 0.2672678980 \ 10^{13} \ L23^3 \ c1^3 - 0.3680410399 \ 10^{15} \ L23 \ c1^4 \ c4$$

$$- 0.1514225992 \ 10^{16} \ L23 \ c1^3 \ c4^2$$

> **eqn3_17_sq:= eqn3_17_sq ;**

$eqn3\_17\_sq := 0.3081571556 \ 10^{16} \ c4^3 + 0.1949337600 \ 10^{10} \ L23^6 \ c4$

$$+ 0.1117043200 \ 10^{10} \ L23^6 - 0.5628934554 \ 10^{10} \ L23^5 + 0.5463853959 \ 10^{16} \ c1$$

$$+ 0.5874245780 \ 10^{16} \ c4^2 + 0.1911253794 \ 10^{16} \ c1 \ c4 \ L23^2$$

$$- 0.5115729560 \ 10^{16} \ L23 \ c1^2 + 0.3470298604 \ 10^{15} \ L23^2 \ c1^3$$

$$- 0.4526879307 \ 10^{13} \ L23^4 \ c1 \ c4 - 0.9760319345 \ 10^{12} \ L23^4$$

$$+ 0.2101294121 \ 10^{15} \ L23^2 + 0.7213238649 \ 10^{16} \ c1^2 - 0.1089761729 \ 10^{16} \ L23$$

$$+ 0.4707956544 \ 10^{16} \ c4 + 0.1339260450 \ 10^{14} \ L23^3 \ c1^2 \ c4$$

$$- 0.2286520320 \ 10^{13} \ L23^4 \ c1^2 \ c4 + 0.1380473656 \ 10^{16} \ L23^2 \ c1^2 \ c4^2$$

$$+ 0.2019950461 \ 10^{16} \ L23^2 \ c1^2 \ c4 + 0.1648149759 \ 10^{16} \ L23^2 \ c4^2 \ c1$$

$$- 0.2693873525 \ 10^{13} \ L23^4 \ c1 \ c4^2 - 0.1095360238 \ 10^{11} \ L23^5 \ c1 \ c4$$

$$- 0.1889778074 \ 10^{16} \ L23 \ c4^3 + 0.1843968000 \ 10^{10} \ L23^6 \ c1 \ c4$$

$$+ 0.7038352097 \ 10^{15} \ L23^2 \ c1^3 \ c4 + 0.2635290684 \ 10^{14} \ L23^3 \ c1 \ c4$$

$+\ 0.1254057344\ 10^{17}\ c1^2\ c4^2 + 0.2975729290\ 10^{16}\ c1^3\ c4^2$

$+\ 0.7563577221\ 10^{15}\ c1\ L23^2 - 0.1062125568\ 10^{13}\ L23^4\ c1^2\ c4^2$

$+\ 0.6050649557\ 10^{16}\ c1^3\ c4 + 0.1721030006\ 10^{14}\ L23^3\ c1\ c4^2$

$+\ 0.8939345721\ 10^{15}\ L23^2\ c1^2 + 0.6747419064\ 10^{13}\ L23^3\ c1^2\ c4^2$

$+\ 0.4953462407\ 10^{13}\ L23^3 + 0.2550625106\ 10^{16}\ c1^2\ c4^3$

$+\ 0.5611375232\ 10^{16}\ c1\ c4^3 - 0.1157952251\ 10^{11}\ L23^5\ c4 + 0.3075746857\ 10^{16}\ c1^3$

$-\ 0.3334902483\ 10^{15}\ c4^4\ L23 + 0.5465625225\ 10^{15}\ c4^4\ c1$

$+\ 0.2779085403\ 10^{13}\ c4^3\ L23^3 + 0.2839097906\ 10^{15}\ L23^2\ c4^3$

$+\ 0.3568741909\ 10^{15}\ L23^2\ c1^3\ c4^2 + 0.1720821738\ 10^{17}\ c1^2\ c4$

$+\ 0.5380397453\ 10^{15}\ L23^2\ c4^3\ c1 + 0.1348633385\ 10^{16}$

$-\ 0.2234879229\ 10^{13}\ L23^4\ c4 - 0.3499544516\ 10^{16}\ L23\ c4^3\ c1$

$+\ 0.5777946668\ 10^{15}\ c4^4 - 0.5324667821\ 10^{10}\ L23^5\ c1$

$-\ 0.4010065920\ 10^{12}\ L23^4\ c4^3 + 0.2628864571\ 10^{13}\ c4^3\ L23^3\ c1$

$-\ 0.1660829642\ 10^{13}\ L23^4\ c4^2 - 0.3154637484\ 10^{15}\ c4^4\ L23\ c1$

$+\ 0.6662224379\ 10^{13}\ L23^3\ c1^2 + 0.6209287703\ 10^{15}\ L23^2\ c4^2$

$+\ 0.1548836731\ 10^{17}\ c1\ c4^2 + 0.1588310692\ 10^{17}\ c1\ c4$

$+\ 0.8354304000\ 10^9\ L23^6\ c4^2 - 0.5789761256\ 10^{10}\ L23^5\ c4^2$

$+\ 0.1172863060\ 10^{14}\ L23^3\ c1 - 0.2066277815\ 10^{16}\ L23\ c1^3$

$-\ 0.2220502329\ 10^{13}\ L23^4\ c1 + 0.1065316071\ 10^{14}\ L23^3\ c4^2$

$-\ 0.4129741774\ 10^{16}\ L23\ c1 - 0.1619380575\ 10^{16}\ L23\ c1^2\ c4^3$

$+\ 0.1075648000\ 10^{10}\ L23^6\ c1 - 0.8361892426\ 10^{16}\ L23\ c1^2\ c4^2$

$-\ 0.1021681926\ 10^{17}\ L23\ c1\ c4^2 + 0.2549101363\ 10^{15}\ L23^2\ c1^2\ c4^3$

$-\ 0.3430626536\ 10^{16}\ L23\ c4 - 0.3890719565\ 10^{16}\ L23\ c4^2$

$+\ 0.7902720000\ 10^9\ L23^6\ c4^2\ c1 - 0.5476801188\ 10^{10}\ L23^5\ c4^2\ c1$

$+\ 0.1289186839\ 10^{14}\ L23^3\ c4 + 0.4812079104\ 10^{14}\ L23^2\ c4^4$

$-\ 0.3793305600\ 10^{12}\ L23^4\ c4^3\ c1 + 0.4551966720\ 10^{14}\ L23^2\ c4^4\ c1$

$-\ 0.1116496308\ 10^{17}\ L23\ c1\ c4 - 0.1221936128\ 10^{13}\ L23^4\ c1^2$

$+\ 0.5952551997\ 10^{15}\ c4\ L23^2 - 0.4127784728\ 10^{16}\ L23\ c1^3\ c4$

$-\ 0.1185533798\ 10^{17}\ L23\ c1^2\ c4 - 0.2061029824\ 10^{16}\ L23\ c1^3\ c4^2$

#+++

> # end

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BIOGRAPHICAL SKETCH

I have extensive educational background in both engineering and science, and also years of work experience with nuclear reactors and its components for the United States Naval Nuclear Reactors. My prior education and experience were in the fields of mechanical engineering, nuclear engineering, mathematics and natural science. In addition, I also had experience as a design and structural analyst of mechanical and nuclear reactor equipment and related components. Furthermore, I worked in maintenance dealing with overhauling the nuclear reactors and their power plant equipment for the United States Naval Nuclear Reactors. I plan to work in academic filed related to my past education and work experience.